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Abstract

This paper analyzes the Japanese government bond (JGB) yield curve using the Black-Gorovoi-Linetsky (BGL) model of interest rates as options with a view to monitoring the JGB market expectations about the Bank of Japan's (BOJ) zero interest rate policy (ZIRP). Main findings are as follows. First, overall fitting performance of the BGL model is much better than that of the original Vasicek model in our sample period from the start of the quantitative monetary easing policy on March 19, 2001 through the end of the ZIRP on July 14, 2006. Second, the shadow interest rate is estimated to be negative throughout the period and rise toward zero quite recently. Third, the first hitting time until the negative shadow interest rate first hits zero shows a very good performance in predicting the ending time of the ZIRP with an error of only about one month. Fourth, the estimated probability density function of the first hitting time shows that the JGB market expectations rapidly converge to the mode value as the ending time of the ZIRP approaches.

JEL Classification: E43, E44, E52, G12

Key Words: Term Structure of Interest Rates, Zero Lower Bound, Options Approach, Shadow Interest Rate, First Hitting Time

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1. Introduction

In this paper, we aim to analyze the Japanese government bond (JGB) yield curve in a zero interest rate environment with a view to monitoring market expectations about monetary policy conducted by the Bank of Japan (BOJ). In particular, we attempt to extract the duration of the zero interest rate policy (ZIRP) expected by the JGB market participants using the Black model of interest rates as options and to evaluate the predictive performance *ex post facto*. The BOJ finally ended the ZIRP on July 14, 2006.

Japan had long suffered from an economic slump since the early 1990s, experiencing deflation in the GDP deflator since 1995 and in the CPI since 1998. According to Krugman [1998], Japan had been in a liquidity trap, a situation where the short-term interest rates hit the zero lower bound. In an attempt to find a breakthrough, the BOJ had responded with (i) a lowering of the overnight uncollateralized call rate to 0.5 percent since the end of 1995, (ii) a further lowering to almost zero percent since 1999, and (iii) the adoption of "quantitative easing monetary policy (QMEP)" with a large expansion of the monetary base. In so doing, the BOJ had also made "a commitment" to maintain this policy package until the core CPI inflation rate registered zero percent or higher on a sustainable basis.[1] To that end, the BOJ actively used various types of market operations including a purchasing operation for long-term JGBs. These policy reactions by the BOJ substantially suppressed medium- and long-term JGB yields, as shown in Figure 1. The BOJ ended the QMEP and ZIRP on March 9 and July 14, 2006, respectively.

This policy challenge had also posed tough challenges to our monitoring of the JGB yield curve in several ways. The first challenge was that it became extremely difficult to trace the real-life shape of the JGB yield curve by the conventional yield curve models. Figure 2 shows the transition

[1] More specifically, under the QMEP, the BOJ provided an ample liquidity supply by using the current account balances (CABs) at the BOJ as the operating policy target. The target for the CABs was raised several times, reaching 30-35 trillion yen in January 2004 compared to the required reserves of approximately 6 trillion yen.

of the JGB yield curve since 1999. Evidently, the flattening of the JGB yield curve, together with an overall downward shift, sufficiently progressed under the ZIRP and QMEP. As a result, conventional models such as the Vasicek or CIR models no longer successfully traced the changing shape of the JGB yield curve.[2] Extremely low levels of the short- and medium-term interest rates reflected the market participants' expectations about the duration of the ZIRP, which had been committed to by the BOJ, particularly in terms of the CPI inflation rate under the QMEP.[3]

The second challenge was how to extract market expectations about the duration of the ZIRP, which may be regarded by the JGB market participants as an indicator of the strength of the BOJ's commitment to maintain the zero interest rate policy. One straightforward way to address this challenge is to use the euroyen futures interest rates.[4] Since the euroyen futures trade future interest rates by definition, the observed prices reflect market expectations about the future path of short-term interest rates. This method was not a panacea, however. For instance, the time horizon of euroyen futures is three years at the longest, so market expectations about the long-run real economy are not likely to be fully priced in. Also, due to speculative trading, euroyen futures interest rates sometimes become very volatile, deviating from their fundamental values.

This paper attempts to tackle these challenges using the Black model of interest rates as options. Black [1995] interprets a nominal short-term interest rate as a call option on the "equilibrium" or "shadow" interest rate, where the option is struck at zero percent. Put differently, Black [1995] argues that the nominal short-term interest rate cannot be negative since currency serves as an option, in that if an instrument should have a negative interest rate, investors choose currency instead. Employing this notion enables us to use an underlying (shadow) spot rate process

[2] See Vasicek [1977] for the Vasicek model, and see Cox, Ingersoll, and Ross [1985] for the CIR model.
[3] As argued by Baba *et al.* [2005], what ZIRP and QMEP intend to do is to "manage expectations."
[4] Another possibility is to use the interest rates of the overnight index swap (OIS). Since the OIS in Japan uses the uncollateralized overnight call rate, the BOJ's target interest rate, its fixed interest rates should reflect market expectations about the BOJ's policy changes more sensitively. Unfortunately, however, we had to wait until the BOJ ended the QMEP for OIS transactions to become active in Japan. See Ooka, Nagano, and Baba [2006] for more details.

that can take on negative values and simply replace all the negative values of the shadow interest rate with zeros for the observed short-term nominal interest rate.

The Black model has the following advantages over other types of models such as a macro-finance model.[5] First, we do not need to assume any *ad-hoc* macro-economic structure. Second, we can significantly improve the fitting to the actual JGB yield curve in the recent very low interest rate environment. Third, we can directly incorporate the notion of "a zero lower bound on the short-term nominal interest rate" in a more straightforward manner. Fourth, we can directly assess the time period until the negative shadow interest rate first hits zero as the expected duration of the ZIRP, as well as the market expectations about the permanent level of the nominal interest rate.[6,7]

While the basic concept of the Black model is quite robust and is appealing particularly for the recent Japanese situation where short-term interest rates have been indeed zero, the model had the disadvantage in that it was analytically intractable.[8] However, Gorovoi and Linetsky [2004] successfully derive the analytical solutions for zero-coupon bonds using eigenfunction expansions under several specifications for the shadow interest rate process. We follow their solutions, and thus we call our Black model the Black-Gorovoi-Linetsky (BGL) model in this paper.

The rest of this paper is organized as follows. Section 2 briefly describes the Black model of interest rates as options, followed by the analytical solution derived by Gorovoi and Linetsky [2004]. Section 3 reports the results of calibrating the Black-Gorovoi-Linetsky (BGL) model to the

[5] Bernanke, Reinhart, and Sack [2004] and Oda and Ueda [2005] estimate the JGB yield curve using a macro-finance approach. Bernanke, Reinhart, and Sack [2004] find that the predicted JGB yield curves lie above the actual yield curves after 1999 and the deviation narrows in November 2000 after the end of the ZIRP, and widens again in June 2001 with the adoption of the QMEP. This result implies that the macro-finance model does not closely trace the actual JGB yield curves.

[6] Further, we do not even need to assume specific distributions for the timing of the policy change.

[7] Black [1995] originally recommends applying his model to the U.S. in the 1930s, during which interest rates were extremely low. On the other hand, Gorovoi and Linetsky [2004] and Baz, Prieul, and Toscani [1998] strongly recommend applying the Black model to the Japanese situation since the late-1990s.

[8] See Rogers [1995, 96] for this line of criticism.

JGB yield curve on a day-to-day basis. Section 4 reports the duration of the ZIRP expected by the JGB market participants and evaluates its predictive performance *ex post facto*. Section 5 concludes the paper. Appendices 1 and 3 report (i) the sensitivity analysis of the yield curve and (ii) the sensitivity analysis of the probability density of the first hitting time to a change in the BGL model parameters, respectively. Appendix 2 refers to the possible extension of the Black model into a two-factor setting.

2. Black Model of Interest Rates as Options and its Analytical Solution

2.1 Black Model

Black [1995] assumes that there is a shadow instantaneous interest rate that can become negative, while the observed nominal interest rate is a positive part of the shadow interest rate. The rationale for this assumption is quite simple: as long as investors can hold currency whose interest rate is zero, nominal interest rates on the other financial instruments must remain non-negative to rule out arbitrage. Specifically, the observed nominal interest rate r_t can be written as

$$r_t = \max\left[0, r_t^*\right] = r_t^* + \max\left[0, -r_t^*\right], \tag{1}$$

where r_t^* is the shadow interest rate. The relationship between r_t and r_t^* is graphically shown in Figure 3. Equation (1) shows that the observed nominal interest rate can be viewed as a call option on the shadow interest rate whose strike price is zero percent. Also, the second equality tells us that the observed nominal interest rate can be expressed as the sum of the shadow interest rate and an option-like value that provides a lower bound for the nominal interest rate at zero percent when the shadow interest rate is negative. We call this option-like value as the floor value in this paper, as in Bomfim [2003]. In other words, the floor has the option to switch investors' funds from bonds to currency, if r_t^* falls below zero.

How should we interpret the shadow interest rate? Let us first present the view of Black

[1995] himself.[9] Suppose the situation where the equilibrium nominal interest rate that clears the savings-investment gap is negative, given a CPI inflation rate. This situation is akin to the so-called liquidity trap, where under deflationary pressures, very low nominal interest rates cause people to hoard money. As a result, it neutralizes monetary policy attempts to restore full employment.[10] However, the prevailing nominal interest rate is zero, since currency exists. The gap between the negative equilibrium interest rate and the prevailing nominal interest rate leaves the recessionary gap uncleared. Examples of such a situation include the United States during the great depression in the 1930s (Bernanke [2002]), and Japan since the mid-1990s (Krugman [1998]). According to Black [1995], the shadow interest rate corresponds to this equilibrium nominal interest rate.

Another possible interpretation focuses on the expected duration during which the shadow interest rate remains negative. As long as the shadow interest rate is negative, JGB market participants think that the ZIRP will continue. Thus, the expected time until the negative shadow interest rate first hits zero is regarded as the expected duration of the ZIRP to the first approximation. In what follows, we follow this second interpretation of ours, but note here that if the JGB market participants thought that the BOJ did not end the ZIRP until the Japanese economy broke out of the liquidity trap, both interpretations are likely to coincide with each other.

Under normal circumstances, r_t^* is sufficiently above zero so that the embedded floor value in equation (1) can be safely ignored. When the short-term nominal interest rates are at zero or near zero, however, long-term interest rates embed more-than-usual premiums in terms of the compounded floor value. In such cases, how can we write the slope of the term structure? First of all, the price of a default-free discount bond with maturity T at time zero, $P(r,T)$, can be expressed under the risk-neutral probability as

[9] Bomfim [2003] and Baz, Prieul, and Toscani [1998] follow this interpretation.
[10] See Keynes [1936] and Hicks [1937] for classical debates about the liquidity trap. For the recent Japanese case, see Baz, Prieul, and Toscani [1998].

$$P(r,T) = \mathrm{E}_r\left[\exp\left\{-\int_{s=0}^{T} r_s ds\right\}\right] = \mathrm{E}_r\left[\exp\left\{-\int_{s=0}^{T} \max\left[0, r_s^*\right] ds\right\}\right], \qquad (2)$$

where $\mathrm{E}_r[\bullet] \equiv \mathrm{E}\left[\bullet \,\middle|\, r_0^* = r\right]$ and the second equality corresponds to the option property of the short-term nominal interest rate.

The slope of the term structure $R(r,T) - r_0$ can thus be interpreted as the value of a portfolio of options since the yield-to-maturity $R(r,T)$ is an average of instantaneous forward rates $f(r,s)$ where $s = 0, \ldots T$, each of which exhibits option properties:

$$R(r,T) - r_0 = \frac{1}{T}\int_{s=0}^{T} f(r,s) ds - \max\left[0, r_0^*\right]. \qquad (3)$$

This is because $f(r,s)$ can be viewed as

$$f(r,s) = \mathrm{E}_r[r_s] + \text{forward premium} + \text{floor value}. \qquad (4)$$

As discount bond prices are derived from forward rates, the option-like floor value is compounded all over the yield curve, resulting in a steeper yield curve than the curve that could be expected should currency not exist.

2.2 Gorovoi and Linetsky's Analytical Solution

2.2.1 Finding an Analytical Solution to the Black Model

The Black model of interest rates as options had the disadvantage that it is analytically intractable. In fact, Rogers [1995, 1996], and Goldstein and Kerstead [1997] criticized the Black model on this basis and favored the models with a reflecting boundary at the zero interest rate, despite the criticism against those models on economic grounds.[11] Gorovoi and Linetsky [2004], however, showed that the Black model is as fully analytically tractable as the reflecting boundary model, and successfully obtained an analytical solution for zero-coupon bonds. Let us briefly review the analytical solution under the risk-neutral probability when the Vasicek model is used for the shadow

[11] Black [1995] argues that when the zero interest rate is a reflecting boundary, the interest rate "bounces off" zero, but this seems strange in terms of a real economic process.

interest rate process:

$$dr_t^* = \kappa\left(\theta - r_t^*\right)dt + \sigma dB_t\,,\;\; r_0^* = r \tag{5}$$

where θ is the long-run level of the shadow interest rate, κ is the adjustment rate toward the long-run level, σ is the volatility, and dB_t is the increment of the standard Brownian motions.

Note that the discount bond price (2) has the form of the Laplace transform of an area functional of the shadow interest rate diffusion:

$$A_t \equiv \int_0^t \max\left[0, r_s^*\right]ds\,. \quad t \geq 0 \tag{6}$$

The area functional measures the area below the positive part of a sample path of the interest rate process up to time *t*. Thus, the discount bond price can be calculated as

$$P(r,T) = \mathrm{E}_r\left[\exp(A_T)\right]. \tag{7}$$

To calculate the price (7), the spectral expansion approach is used. The discount bond price $P(r,T)$ as a function of time to maturity *T*, and the shadow interest rate r, solves the fundamental pricing partial differential equation:

$$\frac{1}{2}\sigma^2 P_{rr} + \kappa(\theta - r)P_r - \max\left[0, r^*\right]P = P_T\,, \tag{8}$$

subject to the initial condition $P(r, 0) = 1$. The solution has the eigenfunction expansion:

$$P(r,T) = \mathrm{E}_r\left[\exp(-A_T)\right] = \sum_{n=0}^{\infty} c_n \exp(-\lambda_n T)\varphi_n(r), \tag{9}$$

$$c_n = \int_{-\infty}^{\infty} \varphi(r)\frac{2}{\sigma^2}\exp\left(-\frac{\kappa(\theta - r)}{\sigma^2}\right)dr\,. \tag{10}$$

Here, $\{\lambda_n\}_{n=0}^{\infty}$ are the eigenvalues with $0 < \lambda_0 < \lambda_1 < \ldots, \lim_{n\to\infty}\lambda_n = \infty$, and $\{\varphi_n\}_{n=0}^{\infty}$ are the corresponding eigenfunctions of the associated Sturm-Liouville spectral problem:[12]

$$-\frac{1}{2}\sigma^2 u''(r) - \kappa(\theta - r)u'(r) + \max\left[0, r^*\right]u(r) = \lambda u(r). \tag{11}$$

From equation (6), we have the following asymptotics for large times to maturity:

[12] We chose 5 for n in the eigenfunction. Gorovoi and Linetsky [2004] argue that for longer times to maturity, several terms are enough to attain accuracy of five significant digits. As time to maturity decreases, more terms have to be added to attain the same accuracy. Considering the trade-off between the accuracy and computational efficiency, we chose 5 for n.

$$\lim_{T\to\infty} R(r,T) = \lim_{T\to\infty}\left(-\frac{1}{T}\ln P(r,T)\right) = \lambda_0 > 0\,. \tag{12}$$

Equation (12) shows that as time to maturity increases, the yield curve flattens out and approaches the principal eigenvalue λ_0. Thus, we can call λ_0 the permanent level of nominal interest rate. It is interesting to observe the time-series movement of the permanent level of nominal interest rate, because it is highly likely to capture the JGB market perceptions about future economic activity, which should be free of the effects of business cycle components and monetary policy. In the Black model of interest rates as options with the solution by Gorovoi and Linetsky [2004], the principal eigenvalue is guaranteed to be strictly positive.[13] In the original Vasicek model, however, the principal eigenvalue can be negative. The negative principal eigenvalue implies that as time to maturity increases, the zero-coupon bond price approaches infinity, which yields a negative asymptotic yield. In what follows, we follow the solution by Gorovoi and Linetsky [2004], and thus call the model the Black-Gorovoi-Linetsky (BGL) model. In Appendix 1, we show how the yield curve is influenced by each of the BGL model parameters: the initial value of the shadow interest rate, θ, κ, and σ.

2.2.2 First Hitting Time until the Negative Shadow Interest Rate First Hits Zero

For policy implications, it is interesting to derive the first hitting time defined as the expected time for the shadow interest rate to become positive again starting from negative values. Specifically, what we would like to estimate is

$$\tau_0 \equiv \min\left[t \geq 0; r_t^* = 0\right]. \tag{13}$$

Linetsky [2004] calculates the probability distribution function (PDF) of the first hitting time for the Vasicek process. In this paper, we use the mode value of the estimated PDF following Linetsky

[13] For more details, see Gorovoi and Linetsky [2004].

[2004] as the representative market expectations of the first hitting time τ.[14]

To calculate the PDF of the first hitting time, Linetsky [2004] uses the eigenfunction expansion approach. Suppose that $r_0^* = r < 0$ and $t > 0$, the PDF of the first hitting time can be written as

$$f_{\tau 0}(t) = \sum_{n=1}^{\infty} d_n \gamma_n \exp(-\gamma_n t), \quad t \geq 0, \tag{14}$$

where $\{\gamma_n\}_{n=0}^{\infty}$ are the eignvalues with $0 < \gamma_0 < \gamma_1 < \ldots < \lim_{n \to \infty} \gamma_n = \infty$, and $\{d_n\}_{n=0}^{\infty}$ are given as

$$d_n = -\frac{H_{\frac{\gamma_n}{n}}\left(\sqrt{\kappa}(\theta - r)/\sigma\right)}{\frac{\gamma_n}{n}\frac{\partial}{\partial \gamma}\left[H_\gamma\left(\sqrt{\kappa}\theta/\sigma\right)\right]\Big|_{\gamma=\frac{\gamma_n}{n}}}, \tag{15}$$

where $H_\gamma(\bullet)$ denotes the Hermite function.[15]

3. Results of Calibrating the BGL Model to the JGB Yield Curve

3.1 Fitting Performance of the BGL Model

This section reports the results of calibrating the BGL model to the JGB yield curve on a day-to-day basis. Throughout the paper, we use the discount bond yields estimated from the price data of coupon bonds with 5-, 10-, and 20-year maturities at issue by McCulloch's [1971, 1990] method.[16,17] The maturity grids we use are 0.5, 1, 2, 3, 5, 7, 10, 15, 18, and 20 years. Thus, we have 10 data points and four parameters including the three Vasichek parameters (θ, κ, and σ) plus the initial shadow interest rate r_0^*. We calibrate the model to the JGB yield curve by minimizing the sum of squared pricing errors between the actual JGB yield curve and the BGL model yield curve on a day-to-day basis.

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[14] The mode value of the first hitting PDF corresponds to the peak of the PDF, which is likely to capture the market consensus about the duration time of the ZIRP.
[15] We use the same procedure for n as in Linetsky [2004]. That is, we use the exact eigenvalues up to n =10, and for more than n =10, we use the asymptotic eigenvalues up to n =200.
[16] In Japan, market liquidity of discount bonds is quite low compared with that of coupon bonds due to the limited issuance.
[17] The data source is the Japan Securities Dealers Association.

First, Figure 4 (i) shows the fitting performance of the BGL model as of February 28, 2005 as a typical example, compared with that of the original Vasicek model. Evidently, the BGL model shows a much better fitting performance, particularly in the short-term maturity zone around 1-2 years. Second, Figure 4 (ii) shows the ratio of the sum of squared errors between the original Vasicek model and the BGL model. It shows that the BGL model performs much better than the original Vasicek model in most of the sample period since the start of the QMEP (March 19, 2001). Exceptions are (i) the first half of 2003, and (ii) since January 2006. According to anecdotal evidence, during these two periods, JGB market participants thought that the ending of the QMEP and ZIRP was imminent.[18] Thus, this comparison reveals that the BGL model shows a much better fitting performance than the original Vasicek model, particularly in the periods during which policy commitment was perceived by the JGB market participants to be strictly binding.[19]

3.3 Time-Series Estimates of the BGL Model Parameters

Figure 5 shows the time-series estimates of the four parameters of the BGL model: (i) the initial value of the shadow interest rate $r_0^* = r$, (ii) the long-run level of the shadow interest rate θ, (iii) the adjustment rate toward the long-run level κ, and (iv) the volatility σ. Noteworthy points are summarized as follows. First, the shadow interest rate is consistently negative under the QMEP and ZIRP. In 2002, it reached less than -15 percent. Since then, it has been on an uptrend, reaching about -2.5 percent when the BOJ finally ended the ZIRP on July 14, 2006.

Second, θ is likely to exhibit a mean-reverting movement. Since September 2001, it had fallen and reached almost zero percent in the middle of 2003, and then it bounced back to about 3

[18] See Nakayama, Baba, and Kurihara [2004] for such anecdotal evidence during the first half of 2003.
[19] From January to June 2003, concerns over global deflation, together with the strong commitment effects of the QMEP, significantly flattened the JGB yield curve. From June to September, however, those concerns eased as the Japanese economic recovery gained momentum with signs that the fall in the CPI inflation rate was coming to an end. These changes shortened the expected duration time of the ZIRP, resulting in a substantial rise in medium-term yields. See Nakayama, Baba, and Kurihara [2004] for more details.

percent by the middle of 2005. After that, it fell to almost zero percent again in May 2006 and reached around 1 percent on July 14, 2006.[20]

Third, the adjustment rate κ shows a somewhat tricky movement. As shown in Appendix 1, κ is closely related to a "twist" between the short-term and long-term yields. When the short-term yields rise relative to the long-term yields, κ tends to rise. Such cases typically correspond to the periods around the middle of 2002.

Fourth, the volatility σ experienced a decline to less than 3 percent around the middle of 2003, and a sharp rise to 6 percent soon after that. Since the middle of 2004, it has been on a gradual downtrend, currently moving around 4 percent. In fact, in the middle of 2003, the JGB yields became very volatile, partly reflecting the upturn of the CPI inflation rate.

3.4 Permanent Level of the Nominal Interest Rate

Next, let us look at the time-series estimates of the permanent level of the nominal rate λ_0 shown in equation (12). For policy makers, it is of particular interest to observe the permanent level of nominal interest rate since it should reflect the market perceptions about the permanent or neutral level of the nominal economic growth rate more properly than the long-run level of the shadow interest rate θ shown in Figure 5(ii). This is because the shadow interest rate should be very sensitive to business cycles, as well as to speculation about the future monetary policy stance rather than the long-run economic growth.

Figure 6 shows the estimated time-series movement of the permanent nominal interest rate, together with the forecasted long-run nominal growth rate calculated as the sum of the real

[20] One interesting extension of the Black model is the extension into the two-factor setting with the long-run shadow interest rate θ as an additional factor. Unfortunately, however, an analytical solution to the two-factor Black model has not yet been found. Appendix 2 reports a "flavor" of the two-factor setting based on the Monte Carlo simulation. The sensitivity analysis suggests that the real driving forces behind the flattening of the yield curve might be a decrease in the volatility of θ (σ_θ) and/or an increase in the adjustment rate toward the long-run level of θ (κ_θ), instead of a decrease in the long-run level of θ ($\overline{\theta}$).

GDP growth rate and the CPI inflation rate forecasted over the next 5-10 years, both of which are published by Consensus Economics as "consensus forecasts." The estimated permanent nominal interest rate shows a very close movement with the forecasted nominal growth rate. In fact, the overall trend of the permanent level of the nominal interest rate seems to be consistent with the anecdotal market observations: the JGB market participants were deeply concerned about the deflationary pressures toward the middle of 2003, and since then they have begun to price in the economic recovery.[21]

4. Monitoring Market Expectations about the BOJ's Monetary Policy

4.1 Estimation Results of the First Hitting Time

In this section, we describe our attempt to monitor market expectations about the timing of the end of the ZIRP using the BGL model. First, let us look at Figure 7(i) and (ii), which exhibit the expected first hitting time τ_0 and the expected ending date of the ZIRP, respectively. Note here that if the long-run level of the shadow interest rate is larger under the risk-neutral probability than under the actual probability, the first hitting time is longer under the actual probability than under the risk-neutral probability reported in Figure 7.[22]

To facilitate comparison, we also show the first hitting times implied by the euroyen futures 3-month interest rates in Figure 7(i). The threshold values of euroyen futures interest rates beyond which we regard as the end of the ZIRP are (i) the date when the euroyen futures interest rate exceeds the average rate when the ZIRP was in place last time (February 1999-August 2000),

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21 See Nakayama, Baba, and Kurihara [2004] for more details.

22 Under the actual probability P, the long-run level of the shadow interest rate θ^P can be written as $\theta^P = \theta + \sigma\lambda/\kappa$ given the constant market price of risk λ. Thus, the relative size between θ^P and θ depends on the sign of λ. When λ is positive (negative), $\theta^P > \theta$ ($\theta^P < \theta$) follows. If $\theta^P > \theta$ ($\theta^P < \theta$) holds, then the first hitting time is longer under the risk-neutral (actual) probability than under the actual (risk-neutral) probability due to the mean-reversion property. Ichiue and Ueno [2006] attempt to distinguish the difference in the first hitting time between under the risk-neutral probability and under the actual probability under the fixed-parameter setting of the BGL model using the monthly data, finding that the difference is small.

and (ii) the date when the euroyen futures interest rate exceeds the average rate when the target for the uncollateralized overnight call rate was 0.25 percent (August 2000-February 2001).

As shown in Figure 7(i), the first hitting time estimated by the BGL model is basically within the band between the two first hitting times implied by the euroyen futures interest rates.[23] This result shows the relevance of the first hitting time estimated by the BGL model as a tool for monitoring market expectations about the BOJ's monetary policy stance. In particular, since around September 2005, the BGL model-implied first hitting time has shown a very close movement to the lower bound of the first hitting time implied by the euroyen futures interest rates and further toward the end of the ZIRP, these three first hitting times almost converged. Also note here that the BGL model-implied first hitting time is about 1.3 months as of July 14, 2006, when the BOJ finally ended the ZIRP. This result implies that the *ex post* predictive error of the BGL model is just one time of the BOJ's monetary policy meeting.

As is evident from Figure 7(ii), another noteworthy point here in retrospect is that during most of the sample period, the first hitting time was underestimated, that is, shorter than the actual time for the BOJ to end the ZIRP. Figuring out reasons behind this underestimation is one of the interesting topics for future research.

4.2 Probability Density Function of the First Hitting Time

Figure 8 displays the PDF of the first hitting time τ_0 for the following four specific dates: (i) the start of the QMEP, (ii) the peak of the QMEP, (iii) the end of the QMEP, and (iv) the end of the ZIRP.[24] From this figure, we can observe the significant changes in the shape of the PDFs across the phases. Specifically, at the peak of the QMEP, the tails of the PDF were very wide, implying

[23] Missing values of euroyen futures interest rates are due to no transactions occurring.
[24] The peak of the QMEP is defined as the date (June 10, 2003) when the first hitting time is estimated to be the longest.

that the market expectations about the end of the ZIRP were highly dispersed. On the contrary, at almost the end of the ZIRP, the tails were much narrower, implying that the market expectations were much more converged to the medium value.

Next, Figure 9 exhibits the time-series movement of the dispersion of the expected first hitting times, calculated as the distance in years between the upper and lower 10 percentile points of the PDFs. From this figure, we can confirm a wide dispersion of the expected first hitting time. Particularly, in 2002, the dispersion was substantially reduced due mainly to a large increase in κ and from around the end of 2002 to 2003, the PDF became much more dispersed due mainly to a decrease in κ and σ. Since early 2005, it has been on a downtrend reflecting a steady increase in the shadow interest rate.[25]

5. Concluding Remarks

This paper has rigorously analyzed the JGB yield curve using the Black-Gorovoi-Linetsky (BGL) model of interest rates as options with a view to monitoring the JGB market expectations about the BOJ's monetary policy during the period from the adoption of the QMEP (March 19, 2001) to the end of the ZIRP (July 14, 2006). Main findings are summarized as follows. First, the overall fitting performance of the BGL model is much better than that of the original Vasicek model during our sample period. Second, the first hitting time shows a very good performance in predicting the ending time of the ZIRP with an error of only about one month. Third, the estimated probability density function of the first hitting time shows that the market expectations rapidly converge to the mode value as the ending time of the ZIRP approaches. These results show that the BGL model did quite a good job in monitoring market expectations under the ZIRP conducted by the BOJ.

[25] See Appendix 3 for the sensitivity analysis of the first hitting time PDF to the BGL model parameters.

References

Baba, N., S. Nishioka, N. Oda, M. Shirakawa, K. Ueda, and H. Ugai [2005], "Japan's Deflation, Problems in the Financial System, and Monetary Policy," *Monetary and Economic Studies*, Bank of Japan, 23, pp.47-111.

Baz, J., D. Prieul, and M. Toscani [1998], "The Liquidity Trap Revisited," *Risk*, September, pp. 139-141.

Bernanke, B. [2002], "Deflation: Making Sure 'It' doesn't Happen Here," Remarks before the National Economists Club, Washington, DC, November 21.

Bernanke, B., V. Reinhart, and B. Sack [2004], "Monetary Policy Alternatives at the Zero Bound: An Empirical Assessment," *Brookings Papers on Economic Activity*, No.2, pp. 1-100.

Black, F. [1995], "Interest Rates as Options," *Journal of Finance*, 50, pp.1371-1376.

Bomfim, A. [2003], "Interest Rates as Options: Assessing the Markets' View of the Liquidity Trap," Working Paper, Federal Reserve Board.

Cox, J., J. Ingersoll, and S. Ross [1985], "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, pp.385-407.

Gorovoi, V., and V. Linetsky [2004], "Black's Model of Interest Rates as Options, Eigenfunction Expansions and Japanese Interest Rate," *Mathematical Finance*, 14, pp.49-78.

Hicks, J. [1937], "Mr Keynes and the Classics: A Suggested Interpretation," *Econometrica*, 5, pp.147-159.

Ichiue, H., and Y. Ueno [2006] "Monetary Policy and the Yield Curve at Zero Interest Rate: The Macro-Finance Model of Interest Rates as Options," Bank of Japan Working Paper Series, No.06-E-16.

Keynes, J. [1936], *The General Theory of Employment, Interest, and Money*, Macmillan, New York.

Krugman, P. [1998], "Japan's Trap," http://www.mit.edu/krugman/www/japtrap.html.

Linetsky, V. [2004], "Computing Hitting Time Densities for OU and CIR processes: Applications to Mean-reverting Models," *Journal of Computational Finance*, 7, pp.1-22.

Merton, R. [1974], "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, 29, pp.449-470.

McCulloch, J. [1971], "Measuring the Term Structure of Interest Rates," *Journal of Business*, 44, pp.19-31.

McCulloch, J. [1990], "U.S. Government Term Structure Data, 1947-1987," in B. M. Friedman and F. Hahn, eds., *Handbook of Monetary Economics* (vol.1), North Holland, pp.672-715.

Nakayama, T., N. Baba, and T. Kurihara [2004], "Price Developments of Japanese Government Bonds in 2003," *Market Review*, 2004-E-2, Financial Markets Department, Bank of Japan.

Oda, N., and K. Ueda [2005], "The Effects of the Bank of Japan's Zero Interest Rate Commitment and Quantitative Monetary Easing on the Yield Curve: A Marco-Finance Approach," CARF Discussion Paper No. CARF-F-013, University of Tokyo.

Ooka, E., T. Nagano, and N. Baba [2006], "Recent Developments of the OIS (Overnight Index Swap) Market in Japan," *Bank of Japan Review Series*, 06-E-4, Bank of Japan.

Rogers, L. [1995], "Which Model for Term Structure of Interest Rates Should One Use?" *Proceedings of IMA Workshop on Mathematical Finance*, IMA, 65, pp.93-116, New York: Springer.

Rogers, L. [1996], "Gaussian Errors," *RISK*, 9, January, pp.42-45.

Vasicek, O. [1977], "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, pp.177-188.

Appendix 1: Sensitivity Analysis of the Yield Curve to a Change in the BGL Model Parameters

Appendix Figure 1 shows the relationship between the shape of the JGB yield curve and each parameter of the BGL model. The baseline case here corresponds to the estimated parameters as of February, 2005 with an additional assumption that the initial value of the shadow interest rate is zero for simplicity. All of the effects turn out to be straightforward. First, an increase in the shadow interest rate shifts upward the whole yield curve almost equally. Second, an increase in θ shifts upward the long-term yields, resulting in a steeper yield curve. This result is intuitive since θ represents the long-run level of the shadow interest rate. Third, an increase in κ shifts upward the short- to medium-term yields slightly, while it shifts downward the long- to super long-term yields simultaneously, which results in a flatter yield curve.[26] Since κ represents the adjustment rate toward the long-run level of the shadow interest rate, a larger value of κ means that the shadow interest rate reverts to its long-run level more quickly, which results in a decrease in risk premiums embedded in the long-term maturity zone. Fourth, an increase in σ shifts upward the overall yield curve, particularly of the long-term maturity zone. This result is also intuitive: the more volatile is the shadow interest rate, the more term premium is added toward the longer maturities.

[26] In fact, the effects of κ on the shape of the yield curve vary depending on the level of other parameters, particularly σ. The result reported in the paper is a typical one.

Appendix 2: Two-Factor Extension of the Black Model

Appendix 2 briefly discusses the implications derived from the two-factor model in which the long-run level of the shadow interest rate θ follows another Vasicek process, in addition to the shadow interest rate itself. The suggested model structure is written as follows:

$$dr_t^* = \kappa_r \left(\theta_t - r_t^*\right)dt + \sigma_r dB_{r,t}$$

$$d\theta_t = \kappa_\theta \left(\overline{\theta} - \theta_t\right)dt + \sigma_\theta dB_{\theta,t},$$

where θ_t is assumed to follow a mean-reverting process with its long-run level $\overline{\theta}$ and the adjustment rate κ_θ. $dB_{r,t}$ and $dB_{\theta,t}$ are uncorrelated increments to the standard Brownian motions for the shadow interest rate and its long-run, respectively, and σ_r and σ_θ are their volatilities. This model is a two-factor extension to the one-factor Black model we use in this paper. An analytical solution has not yet been found for the two-factor Black model, so in this Appendix, we numerically simulate the model using the Monte Carlo simulation method by giving it an arbitrary set of parameters and find how the model works.[27]

Appendix Figure 2 shows the sensitivity analysis of the two-factor Black model. The results can be summarized as follows. First, an increase in the initial value of θ and $\overline{\theta}$ shifts upward the long-term maturity zone. Second, an increase in σ_θ shifts upward the medium- and long-term zones. Third, an increase in κ_θ shifts downward the long-term zone.

The fitting result of the one-factor BGL model needs a fall in θ to capture the flattening of the JGB yield curve. The sensitivity analysis suggests that real driving forces behind the flattening might be a decrease in σ_θ and/or an increase in κ_θ, instead of a decrease in $\overline{\theta}$.

[27] We randomly generate 10,000 samples.

Appendix 3: Sensitivity Analysis of the First Hitting Time PDF to a Change in the BGL Model Parameters

Appendix Figure 3 shows the sensitivity of the first hitting time PDF to the BGL model parameters. The baseline case here corresponds to the estimated parameters as of February, 2005. First, an increase in the shadow interest rate shifts leftward the whole PDF. This is a very intuitive result: the nearer the shadow interest rate to zero percent, the more likely it hits zero percent within a given period of time, other things being equal. Second, an increase in θ slightly shifts the PDF rightward. As shown in Appendix Figure 1, an increase in θ shifts upward the medium- and long-term yields, but exerts smaller effects on the short-term yields. This is likely to be a main reason for the slight shift of the PDF. Third, an increase in κ sharpens upward the whole PDF with almost the same medium value. This is because an increase in κ indicates a more rapid speed of adjustment toward the long-run level of the shadow interest rate, which facilitates the conversion of the expectations about the first hitting time. Fourth, an increase in σ shifts leftward, as well as sharpens upward the whole PDF. This result is also intuitive: the more volatile the shadow interest rate, the more likely it hits zero percent within a given period of time.[28]

Given these properties, let us have another look at Figure 8, together with Figure 5. Now, it is likely that the flattening of the first-hitting PDF from the start to the peak of the QMEP is caused by a substantial decline in σ, and the leftward shift as well as the sharpening upward of the PDF is caused by an approaching of the shadow interest rate toward zero percent.[29]

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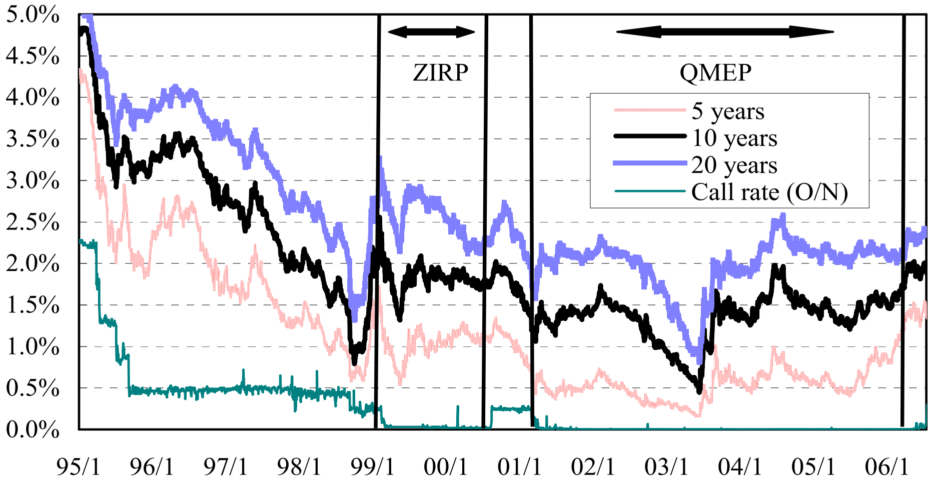
[28] This result is akin to the role of volatility in the Merton [1974] model, in which corporate defaults are models as the first hitting time of the corporate value to debt level.

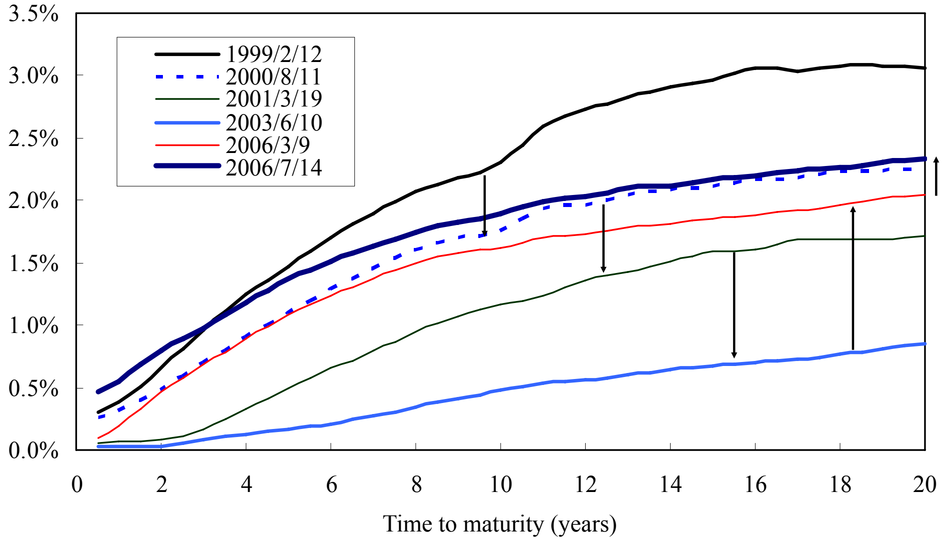
[29] An interesting point here is that the flattening of the PDF from the start to the peak of QMEP is not caused by a decrease in the shadow interest rate.

Figure 1: JGB Yields under ZIRP and QMEP



Note: 5-, 10-, 20-year yields are the discount bond yields estimated from the price data of coupon bonds by McCulloch's [1971, 1990] method. The call rate is the overnight (O/N) uncollateralized call rate.
Sources: Japan Securities Dealers Association, Bank of Japan.

Figure 2: Transition of the JGB Yield Curve



Note: Each date corresponds to the following events. 1999/2/12: the start of the ZIRP; 2000/8/11: the end of the ZIRP; 2001/3/19: the start of the QMEP; 2003/6/10: the peak of the QMEP; 2006/3/9: the end of the QMEP; 2006/7/14: the end of the ZIRP (end of sample period).
Source: Japan Securities Dealers Association.

Figure 3: Shadow and Nominal Interest Rates

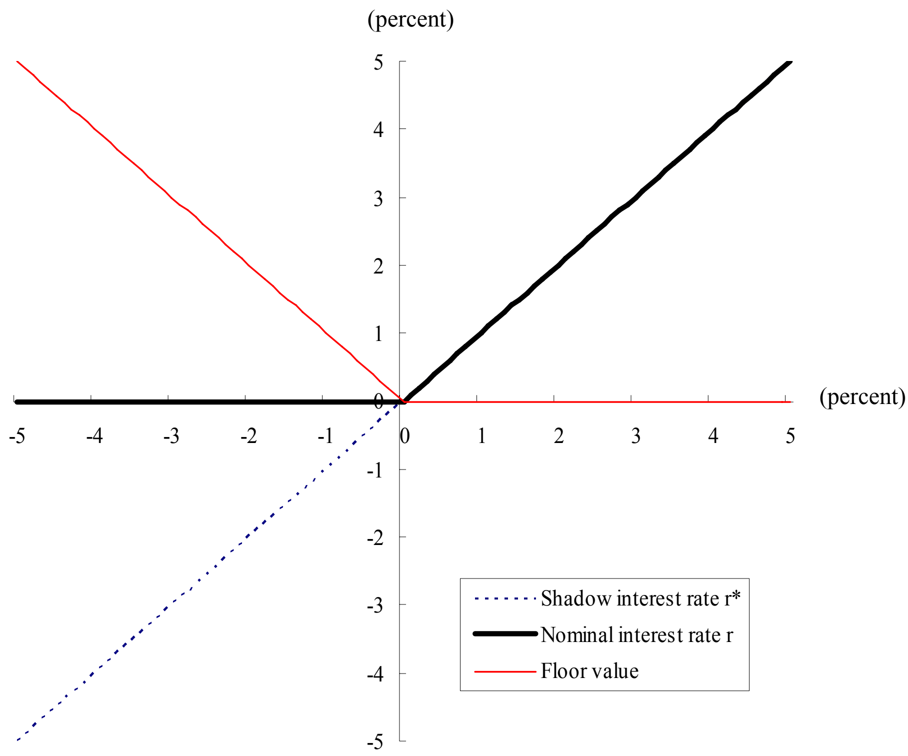
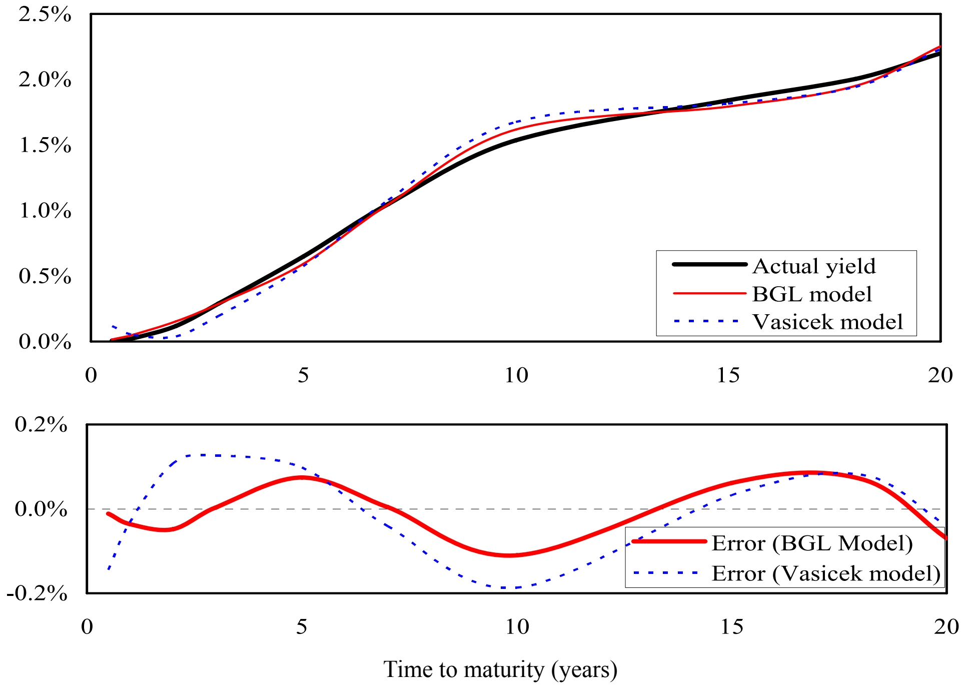


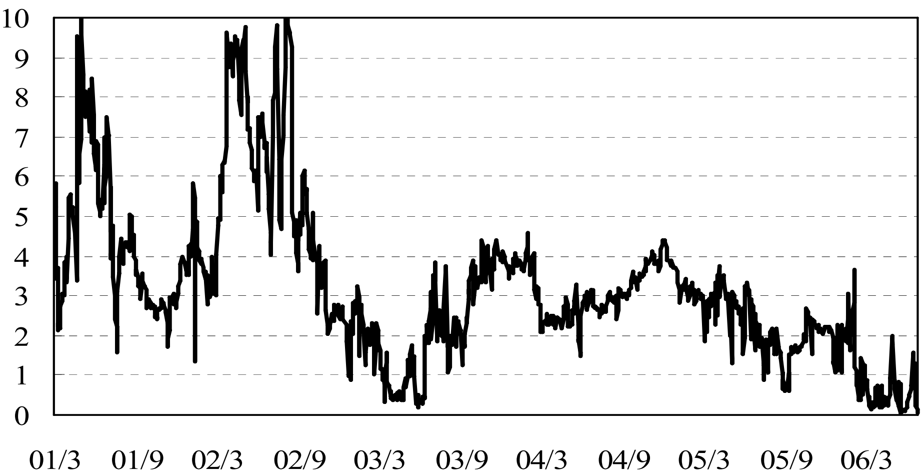
Figure 4: Fitting Performance of the BGL Model to the JGB Yield Curve

(i) Fitting Example of the BGL Model to the JGB Yield Curve



Note: The fitting example is as of 2005/2/28.

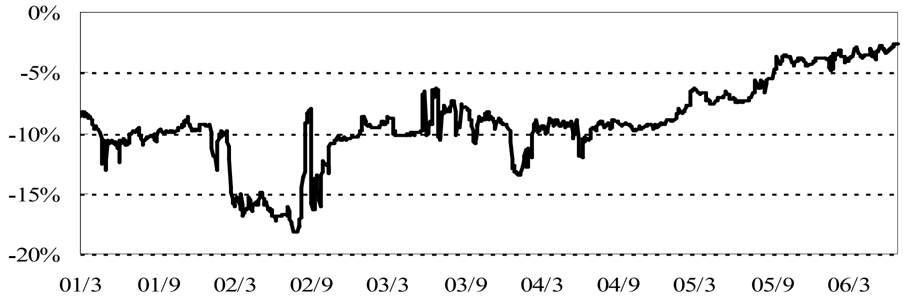
(ii) Ratio of Pricing Errors: Original Vasicek Model vs. BGL Model



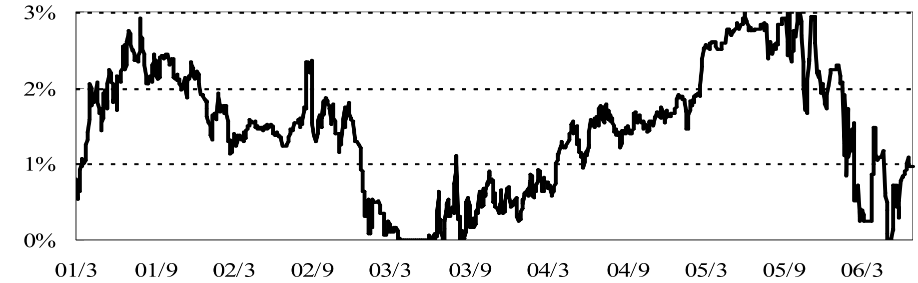
Note: The ratio of pricing errors is calculated by dividing the sum of squared pricing errors from the original Vasicek model by that from the BGL model. The sums of squared errors are for maturities of 0.5, 1, 2, 3, 5, 7, 10, 15, 18, and 20 years.

Figure 5: Estimated Parameters of the BGL Model

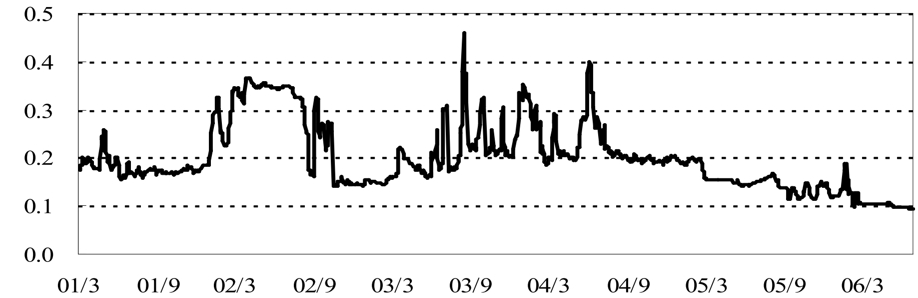
(i) Shadow Interest Rate



(ii) θ : Long-Run Level of the Shadow Interest Rate



(iii) κ : Adjustment Rate toward the Long-Run Level



(iv) σ : Volatility

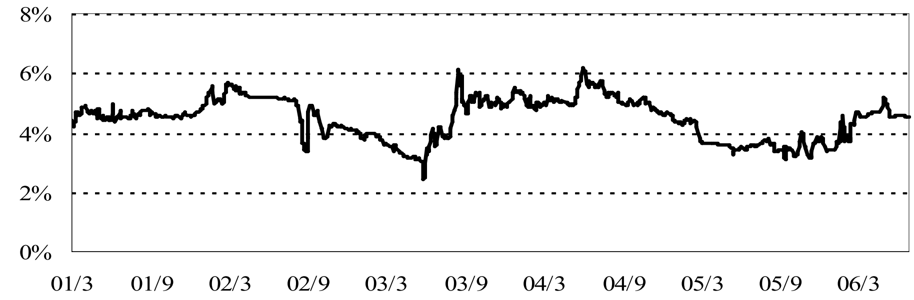
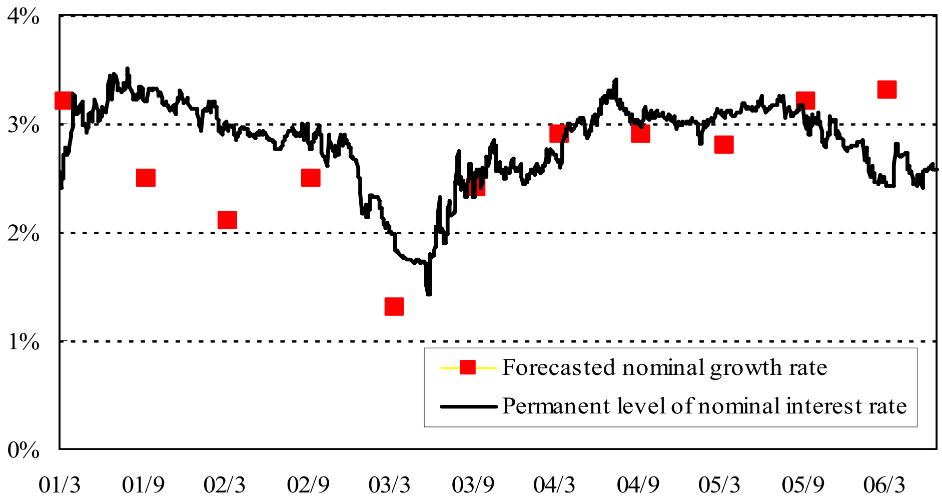


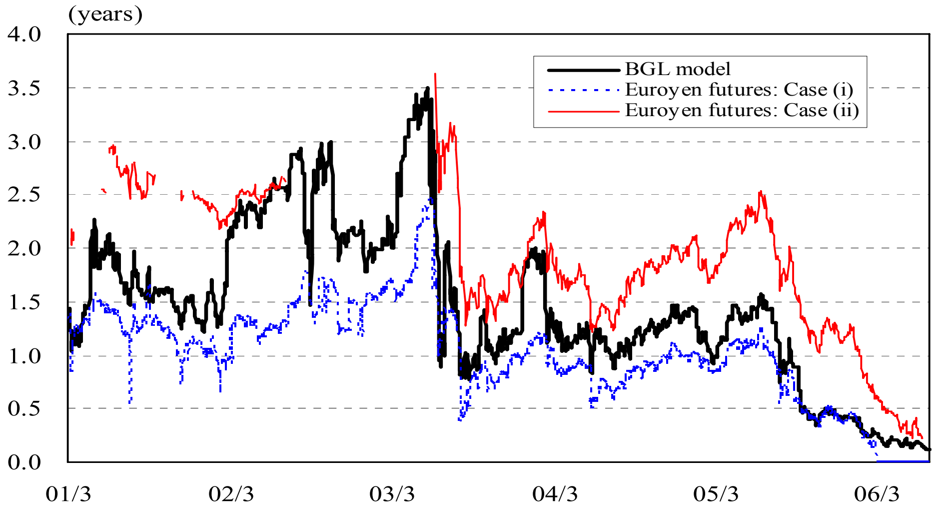
Figure 6: Estimated Permanent Level of Nominal Interest Rate



Note: Forecasted nominal growth rate is the sum of the real GDP growth rate and the CPI inflation rate forecasted over the next 5-10 years.
Source: Consensus forecasts published by Consensus Economics.

Figure 7: First Hitting Time Estimated by the BGL Model

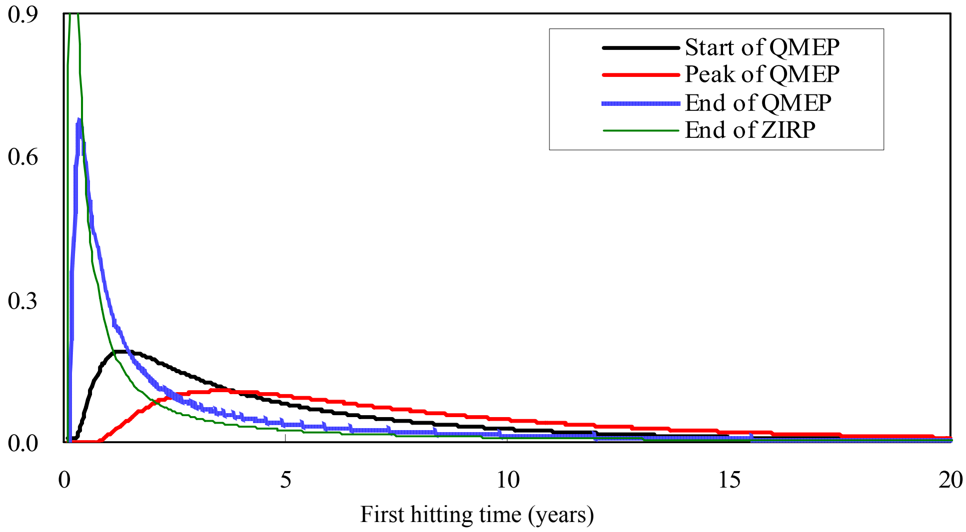
(i) Comparison of the First Hitting Times between the BGL Model and Euroyen Futures Interest Rates



Note: The first hitting times implied by the euroyen futures 3-month interest rates are calculated in the following two cases. Case (i): threshold interest rate is assumed to be 0.19 percent (average of the ZIRP period); Case (ii): it is assumed to be 0.51 percent (average of the period when the target for uncollateralized call rate was 0.25 percent).

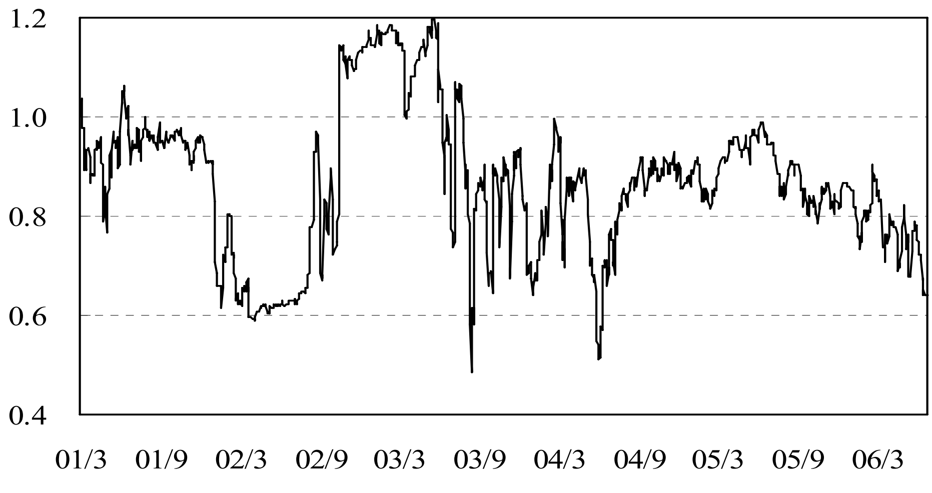
(ii) Expected Ending Date of the ZIRP Estimated by the BGL Model

Figure 8: First Hitting Time PDF Estimated by the BGL Model



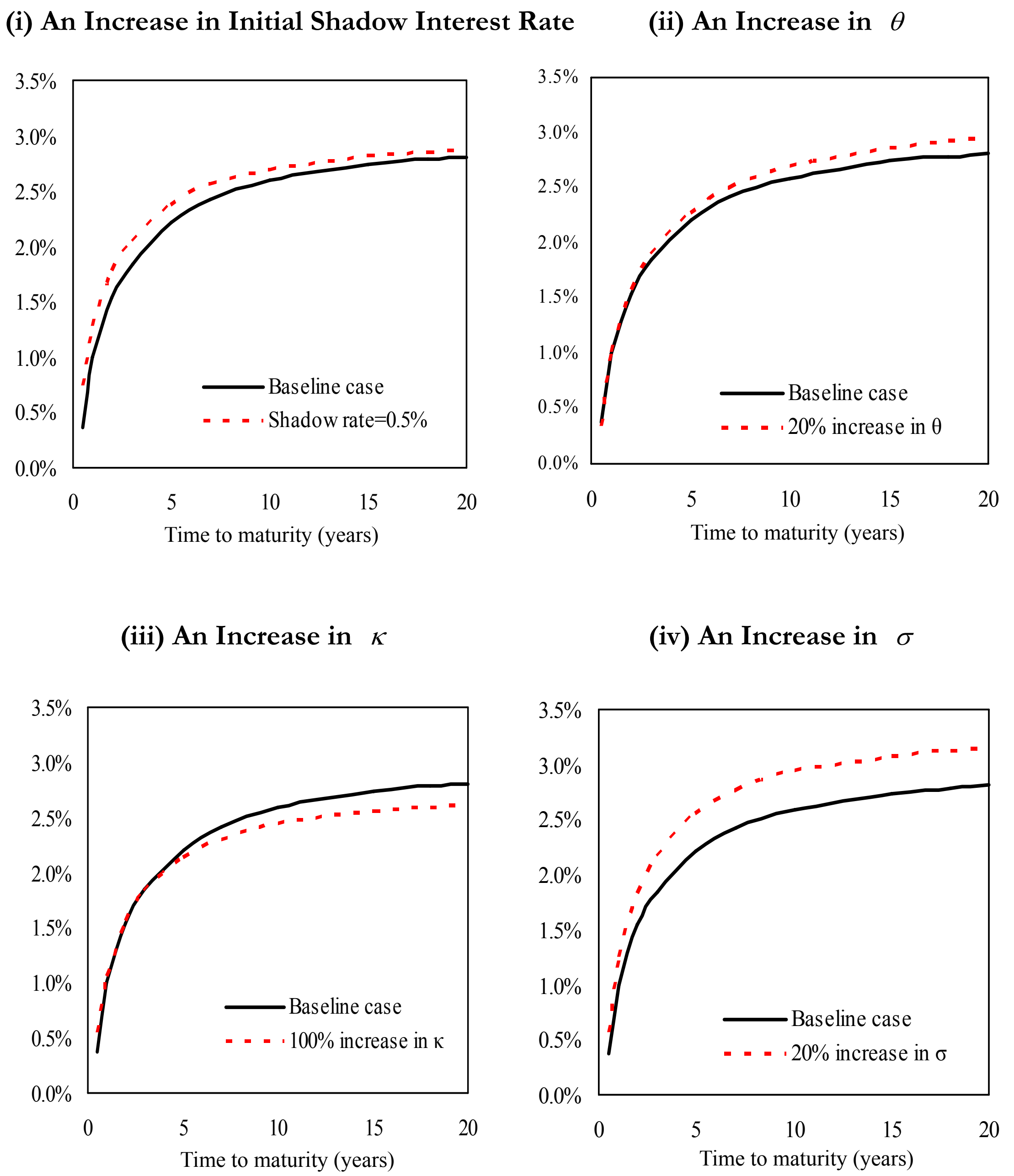
Note: The start of the QMEP is 2001/3/19, the peak of the QMEP is 2003/6/10, the end of the QMEP is 2006/3/9, and the end of the ZIRP is 2006/7/14.

Figure 9: Dispersion of the Expected First Hitting Time



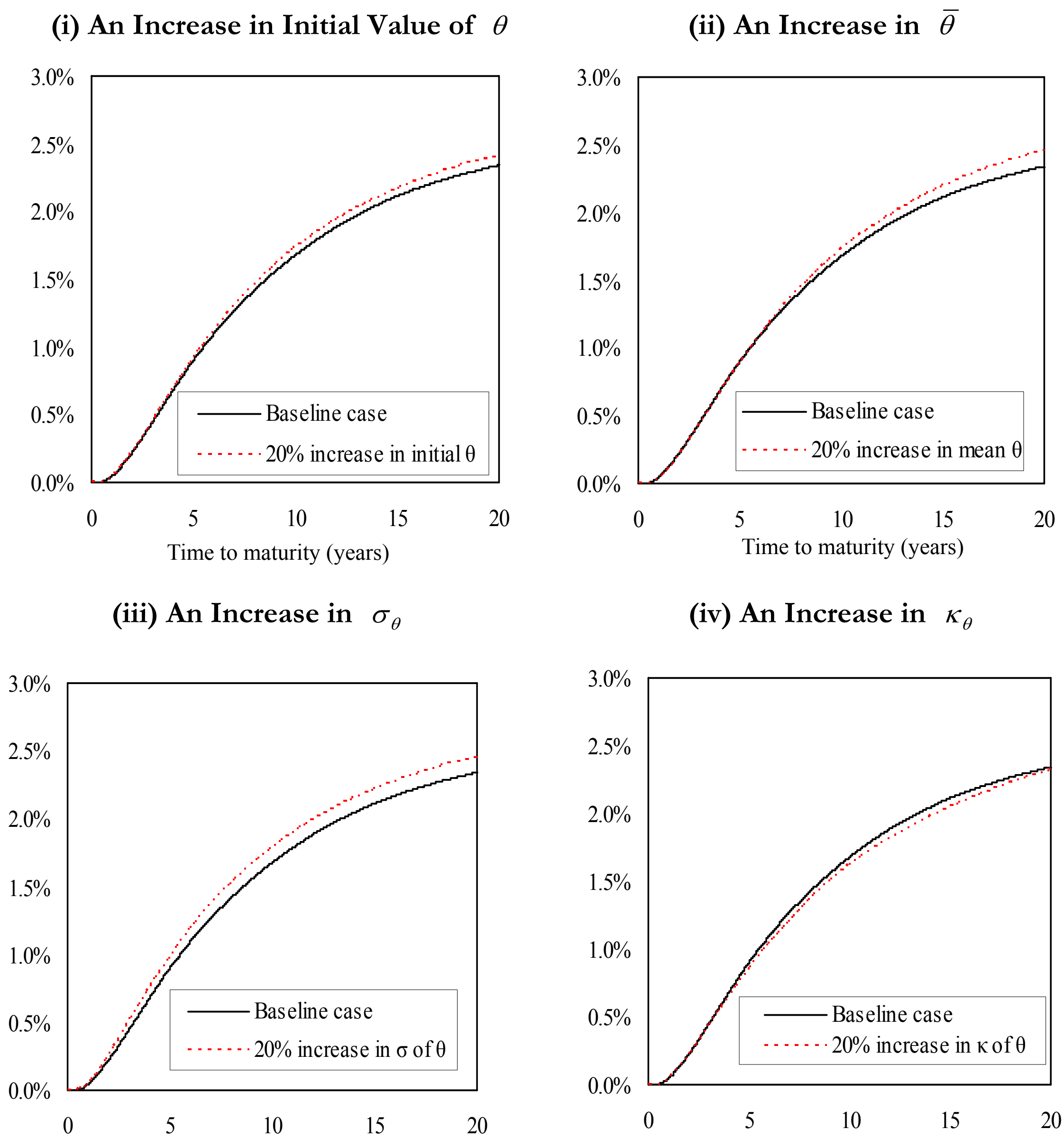
Note: The dispersion of the expected first hitting times is calculated as the distance in years between the upper and lower 10 percentile points of the PDF. The distance is normalized as 2001/3/19=1.

Appendix Figure 1: Sensitivity of the BGL Model to a Parameter Change



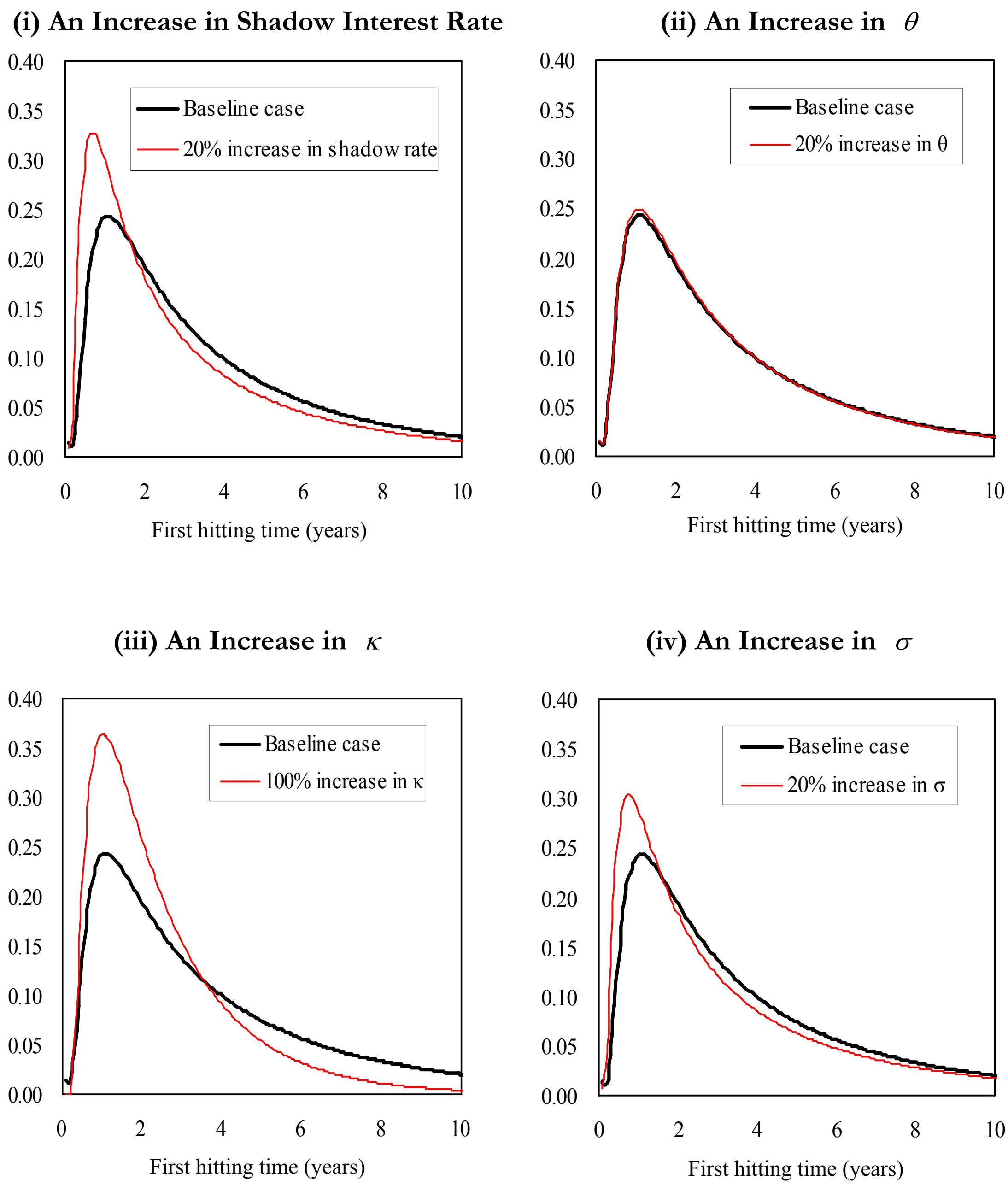
Note: The baseline case is as follows. θ =1.89 percent, κ =0.20, σ =4.37 percent (underlying Vasicek parameters as of 2005/2/28) and the initial value of the shadow interest rate=0.

Appendix Figure 2: Sensitivity Analysis of a Two-Factor Black Model



Note: The baseline case is as follows. κ_r =0.20, σ_r =4.37 percent, the initial value of θ =1.89 percent, the initial value of the shadow interest rate=-7.8 percent, which corresponds to the estimated parameters as of 2005/2/28, κ_θ =0.1, and σ_θ =2 percent. See Appendix 3 for details.

Appendix Figure 3: Sensitivity of First Hitting Time PDF to a Parameter Change



Note: The baseline case is as follows. θ =1.89 percent, κ =0.20, σ =4.37 percent, and the shadow interest rate=-7.79 percent, which correspond to the estimated parameters as of 2005/2/28.