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Monetary Policy and the Yield Curve at Zero Interest: The Macro-Finance Model of Interest Rates as Options

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MONETARY POLICY AND THE YIELD CURVE AT ZERO INTEREST: THE MACRO-FINANCE MODEL OF INTEREST RATES AS OPTIONS[#]

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Abstract

A macro-finance model combined with Black's (1995) model of interest rates as options is employed to investigate the relationship between the yield curve and monetary policy under Japan's zero interest rate environment. The results indicate a strong effect on nominal yields, but not on real yields, under current Bank of Japan policy. This is because the zero rate creates a close link between real yields and expected inflation rates, which are harder to control than expected nominal short rates are. The results also indicate a very flat real yield curve, one that is more stable than the nominal one.

JEL classification: E43; E52

Keywords: Macro-finance, Black's model of interest rates as options, Real interest rates, Monetary policy, Zero interest rate policy

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1. Introduction

The Japanese economy faced serious deflationary pressure for about a decade, starting in the mid-1990s. Japan first experienced negative inflation rates in 1995, measured by the year-on-year percentage change of the consumer price index (CPI; excluding fresh food), for consecutive months, as shown in Figure 1. Then, after a few years with slightly positive inflation rates, Japan had been effectively trapped in deflation until at least 2005.

To counteract the deflationary pressure, the Bank of Japan (BOJ) has adopted various nontraditional policies. First, in 1995, the BOJ lowered the target of the uncollateralized overnight call rate from around 1.75 percent to 0.5 percent, which was the historical low at the time. Market participants concluded that the BOJ could not lower the target rate any further, since that could paralyze the function of the short-term money markets. Second, the BOJ adopted the zero interest rate policy (ZIRP), under which the call rate was lowered to the lowest possible, from February 1999 to August 2000. In April 1999, the BOJ also announced a commitment that the ZIRP would be maintained until the deflationary pressure was dispelled. Third, the BOJ started its quantitative monetary easing policy (QMEP) with another commitment that the QMEP would be maintained at least until the year-on-year percent change of CPI stayed sustainably positive.

These episodes illustrate useful inputs for economists and central banks for deepening the understanding of monetary policy, and have been investigated in the literature. However, our understanding of the policy effects under the zero interest rate is still incomplete. One important reason for this is the difficulty of empirical analyses due to the zero interest rate. That is, the short-term interest rate—the most important monetary policy instrument and conventionally used for the empirical analyses of monetary policy in the other countries—has been bound at zero in Japan. As already mentioned, even 0.5 percent of the call rate could be regarded as bound; thus, the call rate since 1995 should be carefully interpreted.

Yield curve data can be used for empirical analysis on monetary policy, since the time-variation of long-term rates reflects a monetary policy stance even when the short rate is bound at zero. In fact, several papers in the literature try to extract information on monetary policy by using term structure models. These models, however, fail to explain the unusual shape of Japan’s yield curve, which is flatter for shorter maturities; this reflects the expectation that the BOJ will not raise the target rate for a considerable period. For instance, Bernanke, Reinhart, and Sack's (2005) model, in which the expectations for the period of the BOJ's policy are not considered, cannot explain the unusual shape of the yield curve, and thus their empirical results provide limited insights on monetary policy.

To overcome that limitation, we apply Black’s (1995) model of interest rates as options. In the model, the spot nominal rate equals a shadow rate if the shadow rate is positive, and equals zero otherwise. Thus the nominal rate can be interpreted as an option of the shadow rate, and the model takes into account the zero bound of interest rates and the nonlinear relationship between the spot rate and the yield curve. While Black only suggests the idea of the model, Gorovoi and Linetsky (2004) construct the model in detail. Thus the model is also called the Black, Gorovoi, and Linetsky (BGL) model. Gorovoi and Linetsky calibrate a version of their model, in which the shadow rate follows an Ornstein-Uhlenbeck process as in Vasicek (1977), by fitting to the yield curve data at a selected date. They confirm the usefulness of the model for explaining the unusual shape of yield curve under the zero interest rate. Ueno, Baba, and Sakurai (2006) calibrate this Vasicek-type BGL model with their five-year sample since the start of the QMEP and confirm that the model fits the data very well. They also extract the bond market participant's expectations about the duration of the ZIRP.

Our model is a generalization of the Vasicek-type BGL model. Some new aspects of our approach are as follows. First, we estimate the parameter values consistent with time-series data. As already seen, the literature only calibrates the model by fitting to the yield curve data at each point in time independently, and allows time-variation of the parameter values. Since the term structure reflects the predicted future path of the underlying factor, the calibration ignoring the time-series properties of data may result in time-inconsistency and an over-fit problem. On the other hand, to avoid these problems, we attempt to estimate the time-invariant deep parameters of the model. In addition, our approach has another advantage: we can extract the expected duration of the ZIRP or QMEP under the actual probability measure. Typical finance models only provide the duration under risk-neutral measure as an approximation.

Second, we allow time-variation of market prices of risk, which is often observed in empirical studies. Thus the term premia are time-varying in the model, and the variation of the yields can be broken down into those of the expected short rates and term premia. Observing this breakdown enables us to assess quantitatively the relative contribution of the two channels of the monetary policy effect on the yield curve through the expected short rates and term premia. The former channel is termed “the policy duration effect” in the literature, while the other channel may be interpreted as the portfolio rebalance effect.

Third, we employ not only a latent shadow rate but also an observable inflation rate as factors in our macro-finance model. This generalization enables us to examine the relationship among the inflation rate, the yield curve, and monetary policy. In particular, the inclusion of an inflation rate in the model enables us to calculate the term structure of real interest rates. Thus we can evaluate whether the BOJ's policy contributes toward lowering long-term real rates, which can be considered the most important channel for the transmission mechanism of monetary policy.

The rest of this paper is organized as follows. Section 2 briefly reviews our yield-only model, whose sole factor is a shadow rate. This is a simple generalization of a model proposed by Gorovoi and Linetsky. Section 3 describes the data and estimation method, and then shows the estimation results. Section 4 examines the time-series properties of the estimated shadow rate in comparison with monetary policy instruments and the other macroeconomic variables to interpret the empirical results. Section 5 reviews our macro-finance model. We show the model-implied term structure of real interest rates and discuss its properties to derive policy implications. Section 6 concludes the paper.

2. The Yield-Only Model

This section reviews our yield-only model, a generalization of a model proposed in Gorovoi and Linetsky (2004). The nominal spot rate i_t, which is defined as an interest rate per annum, has a zero bound and should be nonnegative. The spot rate is represented as a function of a shadow rate x_t:

$$i_t = \max(x_t, 0)\,. \tag{1}$$

That is, the nominal spot rate is equal to the shadow rate if the shadow rate is positive, and equal to zero otherwise. The shadow rate is assumed to follow an Ornstein-Uhlenbeck process described as

$$dx_t = \kappa(\theta - x_t)dt + \sigma dW_t\,, \tag{2}$$

where $\theta > 0$ is the long-run level of the shadow rate or spot rate, $\kappa > 0$ is the rate of mean reversion toward the long-run level, $\sigma > 0$ is the volatility parameter, and W_t is a standard Brownian motion. The market price of risk λ_t is a linear function of the shadow rate:

$$\lambda_t = \lambda_0 + \lambda_1 x_t \tag{3}$$

with two parameters λ_0 and λ_1. If $\lambda_1 = 0$, our model is identical to Gorovoi and Linetsky's. The nonzero λ_1 allows a time-varying term premia.

Under the no-arbitrage assumption, the T-year nominal discount bond price can be expressed as

$$\exp\{-T \cdot \hat{i}^{(T)}(x_t)\} = \tilde{E}_t[\exp\{-\int_t^{t+T} i_s ds\}]\,, \tag{4}$$

where $\hat{i}^{(T)}(x_t)$ is the continuously compounded T-year nominal discount rate and $\tilde{E}_t[\cdot]$ is the conditional expectation under the risk-neutral measure. The Ornstein-Uhlenbeck process (2) can be rewritten as

$$dx_t = \tilde{\kappa}(\tilde{\theta} - x_t)dt + \sigma d\tilde{W}_t, \tag{5}$$

where $\tilde{\kappa}\tilde{\theta} = \kappa\theta - \sigma\lambda_0$, $\tilde{\kappa} = \kappa + \sigma\lambda_1$, and $d\tilde{W}_t = dW_t + \lambda_t dt$ hold, and $\tilde{W}_t$ is a standard Brownian motion under the risk-neutral measure. Note that the risk-adjusted long-run level of the shadow rate $\tilde{\theta}$ is approximately interpreted as the infinite maturity rate. To solve (4) for obtaining the model-implied nominal rates $\hat{i}^{(T)}(x_t)$ given x_t and parameter values, we use the closed form solution derived by Gorovoi and Linetsky.[1] Although our model is generalized with nonzero λ_1, their approach can be applied by substituting their κ with $\kappa + \sigma\lambda_1$.

3. Estimation of the Yield-Only Model

This section considers the data and estimation. Gorovoi and Linetsky (2004) calibrate their model only at a selected date. This kind of approach—in which the parameters are chosen to fit the data only at each point in time independently—ignores

[1] The closed form solution is represented with an infinite sum. Thus, in practice we should use a partial sum, and the results are just approximations. To confirm the robustness of the results, we also used a lattice method and obtained very similar results.

the time-series properties of the data. To overcome the limitation of that approach, we use the Kalman filter to estimate the parameters. Subsection 3.1 describes the data and estimation methods, and subsection 3.2 shows the estimation results.

3.1 Data and Estimation Methods

We use end-of-month data over the period from January 1995 to March 2006.[2] The collateralized overnight (O/N) call rate is used as the proxy for the spot nominal rate.[3] We also use 0.5-, 2-, 5-, 10-year maturity zero-coupon yield data[4].

For estimating our continuous-time model with the discrete-time data, the model should be discretized. The discretized shadow rate process is written as the following AR(1) process:

$$x_t = \mu + \phi x_{t-1} + \sigma_\eta \eta_t \,, \tag{6}$$

where $\mu = \theta(1-\phi)$, $\phi = e^{-\kappa/12}$, $\sigma_\eta = \sigma\sqrt{(1-e^{-\kappa/6})/2\kappa}$, and $\eta_t \sim N(0,1)$ i.i.d. Since the shadow rate is unobservable when the spot rate is bounded at zero, the Kalman filter is employed to estimate the parameters and the shadow rate process. In using the Kalman filter, conditional linearization is needed. Equation (1) can be linearized as $i_t = 1_{\{x_{t|t-1} \geq 0\}} x_t$,

[2] The estimation of Black's model tends to be influenced by the data of the middle-term rates such as 5-year maturity yield, because these rates are relatively sensitive to the shadow rate when the shorter maturity rates are zero. The Japanese government started to increase markedly the share of the issuance of the middle-term bonds around 1995, which contributed to the higher liquidity. That is why we choose the sample period from 1995.

[3] In fact, the Bank of Japan uses the uncollateralized call rate as a policy instrument. However, we use the collateralized rate, which is related less with the credit conditions of the market participants and is more appropriate for considering the relationship with risk-free government bond yields.

[4] The yield data are constructed using methods proposed by McCulloch (1990).

where $1_{\{x_{t|t-1}\geq 0\}}$ equals one if the optimal forecast of x_t at $t-1$, $x_{t|t-1}=\mu+\phi x_{t-1}$, is nonnegative, and zero otherwise. The observed O/N call rate $i_t^{\text{O/N}}$ is assumed to equal the spot rate with an error $\varepsilon_t^{\text{O/N}}$:

$$i_t^{\text{O/N}} = 1_{\{x_{t|t-1}\geq 0\}} x_t + \varepsilon_t^{\text{O/N}}. \tag{7}$$

The model-implied *T*-year yields $\hat{i}^{(T)}(x_t)$ are also conditionally linearized and the observed yields $i_t^{(T)}$ are assumed to equal the model-implied ones with errors $\varepsilon_t^{(T)}$:

$$i_t^{(T)} = \hat{i}^{(T)}(x_{t|t-1}) + \hat{i}^{(T)}{}'(x_{t|t-1})\cdot(x_t - x_{t|t-1}) + \varepsilon_t^{(T)}, \tag{8}$$

for $T = 0.5$, 2, 5 and 10[5]. The differentiated model-implied rate $\hat{i}^{(T)}{}'(x_{t|t-1})$ can be calculated using the Gorovoi and Linetsky's closed-form solution. The errors $(\varepsilon_t^{\text{O/N}}, \varepsilon_t^{(0.5)}, \varepsilon_t^{(2)}, \varepsilon_t^{(5)}, \varepsilon_t^{(10)})'$ are assumed to follow serially and mutually uncorrelated normal distributions.

In sum, we estimate five parameters in the model: κ, θ, σ, λ_0, and λ_1. These estimates are used to obtain the implied values for $\tilde{\theta}$, $\tilde{\kappa}$, μ, ϕ, and σ_η. The maximum likelihood estimation is applied for the estimation.

[5] Duffee (1999) uses a similar Kalman filter with conditional linearization for his defaultable bond pricing model.

3.2 Estimates and Model Fit

Table 1 reports the estimates of the five parameters and the implied value of the infinite maturity rate $\tilde{\theta}$ with the standard errors. The estimates of the long-run levels of the spot rate and the infinite maturity rate are 1.45 percent and 3.89 percent respectively. Thus the long-run level of term spread between them is 2.44 percent. Let us discuss the parameters for discrete-time dynamics μ, ϕ and σ_η in Table 2, instead of those for continuous-time dynamics κ and σ, for helping intuition. These parameters are found to be very similar to the OLS results for the call rate over the sample from January 1980 to December 1994. This suggests that the estimates are reasonable, because the call rate is a proxy for the shadow rate when the shadow rate is positive, and thus their time-series properties should be similar. Another notable result is the estimate of λ_1, which is positive but insignificantly different from zero. This means that the term premium is suppressed when the shadow rate declines, although the reduction in the premium is statistically insignificant.

Table 3 reports the standard deviations of the residuals $\varepsilon_t^{\mathrm{O/N}}$, $\varepsilon_t^{(0.5)}$, $\varepsilon_t^{(2)}$, $\varepsilon_t^{(5)}$ and $\varepsilon_t^{(10)}$, and Figure 2 shows the residual for the estimation of 5-year maturity yield and the average yield curve for 2002-03 when the yield curve was the most flattened. These results suggest that the model fits the data well particularly for the short to middle term yields. That is, our model successfully replicates the yield curve shape with a flatter slope for short to middle maturities, which is observed when the short rate is very close to zero, and is difficult to replicate by conventional models.

4. The Shadow Rate

This section considers the implications of the estimated yield-only model. We focus on the estimated shadow rate, which provides very useful insights. We observe the shadow rate in comparison with policy and economic variables in subsections 4.1 and 4.2. Subsection 4.3 considers what the shadow rate means and how it influences the yield curve.

4.1. Monetary Policy Instruments and the Shadow Rate

Figure 3 shows the overnight call rate and the quantitative money target by the BOJ in comparison with the shadow rate. The shadow rate fell into the negative region in September 1995 and has remained there since June 1997, even when the call rate moved around 0.5 percent and before the BOJ commenced the ZIRP in February 1999. This result implies that the bond market participants regarded a 0.5 percent call rate as zero percent in effect. In fact, the shadow rate was negative when the BOJ terminated the ZIRP and raised the target of call rate to around 0.25 percent. This result supports our view that a call rate lower than 0.5 percent has been indistinguishable from the zero interest rate by market players. Panel (b) also gives us a hint in considering the effect of the QMEP conducted by the BOJ. The shadow rate had sharply declined around the start of the QMEP in March 2001, and did so again when the BOJ raised its quantitative money target from 10-15 to 27-32 trillion yen in 2002-03, and finally reached -6.8 percent on May 2003.[6] These findings seem to support the strong effect of the QMEP.

[6] Some readers may wonder whether -6.8 percent is too low. Note, however, the 10-year maturity yield declined to only 0.6 percent then. This means that the average expected short rates for the next 10 years should be less than 0.6 percent, if the term premium is positive, as often observed. To satisfy this condition, a sufficient distance to 0 percent is needed and thus -6.8 percent is not so low. In fact,

Another noteworthy point here, however, lies in the recent shadow rate. The shadow rate has risen since 2003 even though the BOJ keeps its quantitative money target at 30-35 trillion yen. This suggests that another factor is needed to explain the behavior of the shadow rate.

4.2. Macroeconomic Variables and the Shadow Rate

Figure 4 shows macroeconomic variables, the inflation rate, and the output gap, in comparison with the shadow rate. This shows a strikingly close link between the shadow rate and inflation rate, but a much weaker link between the shadow rate and the output gap. The shadow rate follows the inflation rate with a lag of six months on average, at which the correlation reaches 0.65. Thus we can conclude that the inflation rate is the most important factor for driving the shadow rate.

Table 4 reports five regression results for confirming the link between the shadow rate and inflation rate. The first regression, in which the shadow rate is regressed on a constant and the inflation rate, suggests that the shadow rate responds to the inflation rate significantly. The second and third regressions, however, suggest that using the call rate instead of the shadow rate provides a different implication. The call rate is significantly less related to the inflation rate after 1995, obviously because the BOJ has faced the zero bound of interest rates and could not respond to the decline in the inflation rate. The fourth and fifth regressions are modified from the first and second ones by adding a lag of the dependent variables. These also suggest a stronger link between the shadow rate and the inflation rate. This link may be interpreted as the monetary policy

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Panel (a) in Figure 3 supports our argument that the shadow rate has similar time-series properties as the call rate, and -6.8 percent in May 2003 is quite natural.

rule implicitly committed by the BOJ, as will be discussed in the next subsection.

4.3. The Policy Duration Effect and Term Premia

From December 2005 to January 2006, inflation jumped up from 0.1 percent to 0.5 percent[7]. Following this jump, market players began to predict a quicker end of the ZIRP[8] than previously expected, and the middle-term yields started to rise sharply, which was associated with a sharp rise in the estimated shadow rate. This finding suggests that the shadow rate is somehow linked to the duration of the ZIRP. Following Ueno, Baba, and Sakurai (2006), we calculate the time until the shadow rate first hits zero (first hitting-time) and interpret this as the duration. Panel (a) of Figure 5 shows the probability density functions of the first hitting-time as of the end of March 2003, when the shadow rate reached the minimum, and March 2006, the last date in our sample. Evidently, the probability density function has shifted leftward, which implies the duration has become shorter. The mode, which corresponds to the peak of the probability density function and which can be interpreted as the main scenario of the duration[9], has shifted from 3.68 to 0.21 years.[10] Panel (b) exhibits the time-variation of the main scenario for the ending time of the ZIRP. The gap between the expected end of the ZIRP and the 45 degree line is the expected duration, and is negatively correlated with the shadow rate, which supports

[7] The inflation rate is very persistent and has seldom changed by more than 0.4 percent per month.

[8] When the BOJ terminated the QMEP in March 2006, it adopted a policy similar to the ZIRP. Since the new policy is not different from the ZIRP in the sense that the BOJ controls the call rate to be as low as possible, we do not distinguish between them.

[9] The mode is better than the mean for comparing with surveys of economic forecasts, which can be interpreted as the main scenarios among market participants.

[10] In the most typical scenario, the BOJ was expected to raise the target rate to 0.25 percent when the ZIRP was terminated. Thus the duration to when the short rate hits 0.25 percent may be better for our purposes. This type of main scenario of the duration as of the end of March 2006 was 0.29 years, which corresponds to July 2006. In fact, the BOJ terminated the ZIRP on July 14th.

our interpretation on the shadow rate. These results are consistent with Ueno, Baba, and Sakurai (2006), although they use a different model and estimation method.[11]

In sum, the shadow rate corresponds to the duration of the ZIRP, through which the yield curve shape is driven in that a more negative shadow rate—and thus a longer duration of the ZIRP—result in a flatter yield curve. Even when the short rate was bound, the BOJ successfully controlled the shadow rate by its commitment on the link between the policy duration and inflation, by which the market participants can believe that the BOJ does not raise the short rate until inflation reaches at least zero percent.

Let us consider how the BOJ's policy has influenced the yield curve. To that end, we define the term premium as[12]

$$TP_t^{(T)} = \frac{1}{T}\log\left(E_t\left[-\int_t^{t+T} i_s ds \right] / \tilde{E}_t\left[-\int_t^{t+T} i_s ds \right] \right). \tag{9}$$

With this definition, we can break down the estimated yield into an expectation component and term premium. Figure 6 seems to show that either the expectation component or term premium has contributed to the downtrend of the 5-year maturity yield up to 2003, and to the uptrend since then. In fact, the time-variation of the expectation component is larger, which implies that this plays a more important role than the term premium does. This result suggests that the policy duration effect—in which the commitment on the duration of zero interest rate influences the yield curve through the

[11] The first hitting-time should be calculated under actual probability measure as done here. However many studies including Ueno et al. in the finance literature provide hitting-times only under risk-neutral measure, which can be pointed out as their limitation.

[12] Although this definition of term premium is mathematically convenient, this is different from the most typical definition by a Jensen term. We computed the typical term premium to confirm the difference is negligible.

expected short rates—works well. Recall that the estimate of λ_1 is not significantly different from zero, and thus the time-variation of the term premium is statistically insignificant. Thus, the other effects through the term premium, such as the portfolio rebalance effect, are not statistically confirmed.

5. The Macro-Finance Model and Real Term Structure

The previous section revealed the strong link between the shadow rate and the inflation rate. This section therefore employs a two-factor macro-finance model with these variables. This model enables us to calculate the term structure of real interest rates, which matter to the real economy in standard economic theory, something that the monetary policy authorities should pay attention to, rather than to the nominal yield curve. Subsection 5.1 reviews the model. Subsection 5.2 describes the data and calibration procedure. Subsection 5.3 shows the empirical results and discusses the properties of real term structure. In particular, we're concerned about whether or not the BOJ's policy successfully lowered the real term structure, when the Japanese economy faced serious deflationary pressure.

5.1. The Macro-Finance Model

The model has two-factors: an inflation rate π_t and shadow rate x_t. The shadow rate has exactly the same definition as in the yield-only model, and is linked to the spot nominal rate by equation (1). The factors follow a vector Ornstein-Uhlenbeck process,

$$d\mathbf{z}_t = \mathbf{K}(\boldsymbol{\theta} - \mathbf{z}_t)dt + \boldsymbol{\Sigma} d\mathbf{W}_t \tag{10}$$

where $\mathbf{z}_t = (\pi_t, x_t)'$ and $\boldsymbol{\theta} = (\theta_\pi, \theta_x)'$ are the factor vector and its long-run level respectively, $\mathbf{K}$ corresponds to the rate of mean reversion toward the long-run level, and $\mathbf{W}_t = (W_{\pi,t}, W_{x,t})'$ consists of the two orthogonal standard Brownian motions. For our analysis, the covariance matrix of the Brownian motion term $\boldsymbol{\Sigma\Sigma}'$ matters, and we do not have to identify all four parameters in $\boldsymbol{\Sigma}$. Thus without losing generality, we assume that $\boldsymbol{\Sigma}$ is lower-triangle. The market-price of risk depends on the factors:

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1 \mathbf{z}_t , \tag{11}$$

for a 2×1 vector $\boldsymbol{\lambda}_0$ and a 2×2 matrix $\boldsymbol{\lambda}_1$.

Under the no-arbitrage assumption, the *T*-year nominal discount bond price can be calculated with the same equation as for the yield-only model,

$$\exp\{-T \cdot \hat{i}^{(T)}(\mathbf{z}_t)\} = \tilde{E}_t[\exp\{-\int_t^{t+T} i_s ds\}] , \tag{12}$$

although the model-implied long-term rates depend on inflation rate π_t other than the shadow rate x_t, and the factor process under the risk-neutral measure is represented as

$$d\mathbf{z}_t = \tilde{\mathbf{K}}(\tilde{\boldsymbol{\theta}} - \mathbf{z}_t)dt + \boldsymbol{\Sigma} d\tilde{\mathbf{W}}_t , \tag{13}$$

where $\tilde{\mathbf{K}}\tilde{\boldsymbol{\theta}} = \mathbf{K}\boldsymbol{\theta} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0$, $\tilde{\mathbf{K}} = \mathbf{K} + \boldsymbol{\Sigma}\boldsymbol{\lambda}_1$, $d\tilde{\mathbf{W}}_t = d\mathbf{W}_t + \boldsymbol{\lambda}_t dt$, and $\tilde{\mathbf{W}}_t$ is the mutually uncorrelated vector standard Brownian motion under the risk-neutral measure. Given the parameter values and the factor vector $\mathbf{z}_t$, equation (12) can be solved to obtain $\hat{i}^{(T)}(\mathbf{z}_t)$ by applying a lattice method.[13] Totally, we need to calibrate 15 parameters in $\boldsymbol{\theta}$, $\tilde{\boldsymbol{\theta}}$, $\mathbf{K}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{\lambda}_1$, with which the others can be determined.

5.2. Data and Calibration

We use monthly data over the period from January 1995 to March 2006 ,as is the case with the yield-only model. The year-on-year percent change of the CPI (excluding fresh food, the consumption tax change in 1997 adjusted) is used as the inflation rate factor. As the other factor, we use the shadow rate estimated for the yield-only model as if the shadow rate is an observable variable. Obviously, estimating the shadow rate freely just for the two-factor model is more efficient. With only an 11-year sample and the model with 15 parameters, however, efficient estimation easily results in an over-fit problem. In addition, using the common shadow rate is helpful for interpreting the results from the macro-finance model consistently with those from the yield-only model

The parameters are calibrated as follows. In the first step, the long-run levels of the shadow rate and the infinite maturity rate are calibrated using the estimation results for the yield-only model as $\theta_x = 1.45\%$ and $\tilde{\theta}_x = 3.89\%$. The long-run level of the inflation rate is chosen as $\theta_\pi = 1\%$ by referring the understanding of the BOJ.[14] These

[13] See, for example, Amin and Bodurtha (1995) for lattice methods. Note that Gorovoi and Linetsky (2004) propose closed form solutions only for one-factor models.

[14] According to the BOJ's announcement in March 2006, the level of inflation rate that each Policy

calibrations to the long-run levels are essential, because the full estimation with our sample—in which the average inflation rate and shadow rate are negative—easily yields unrealistic parameter values with negative long-run levels. In the second step, $\mathbf{K}$ and $\mathbf{\Sigma}$ are estimated using a restricted VAR. The vector Ornstein-Uhlenbeck process (10) can be discretized to a VAR(1) form:

$$\mathbf{z}_t - \boldsymbol{\theta} = \mathbf{\Phi}(\mathbf{z}_{t-1} - \boldsymbol{\theta}) + \mathbf{\Sigma}_\eta \boldsymbol{\eta}_t \,, \tag{14}$$

where $\mathbf{\Phi}$ is a 2×2 matrix and $\mathbf{\Sigma}_\eta$ is a 2×2 lower-triangle matrix, which are functions of $\boldsymbol{\theta}$, $\mathbf{K}$ and $\mathbf{\Sigma}$. Thus given $\boldsymbol{\theta}$ calibrated in the first step, $\mathbf{K}$ and $\mathbf{\Sigma}$ can be determined by the estimates of VAR(1) without constant terms for the mean-adjusted factor vector $\mathbf{z}_t - \boldsymbol{\theta}$. Finally, $\tilde{\theta}_\pi$ and $\boldsymbol{\lambda}_1$ are chosen by minimizing the sum of squared fitting errors of the model:

$$\min_{\{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1\}} \sum_{t=1}^{T} \{(\hat{i}_t^{(0.5)} - i_t^{(0.5)})^2 + (\hat{i}_t^{(2)} - i_t^{(2)})^2 + (\hat{i}_t^{(5)} - i_t^{(5)})^2 + (\hat{i}_t^{(10)} - i_t^{(10)})^2\} \,. \tag{15}$$

This minimization results in an interesting parameter value of the risk-adjusted long-run level of the inflation rate as $\tilde{\theta}_\pi = 2.75\%$, which implies that the infinite maturity real rate approximately equals $\tilde{\theta}_x - \tilde{\theta}_\pi = 1.14\%$. Since the long-run level of the spot real rate

Board member understands as price stability from a medium- to long-term viewpoint, in the conduct of monetary policy ("an understanding of medium- to long-term price stability"), was discussed, and most Board members' median figures fell on both sides of one percent. The one percent seems to be not far from the market consensus on the long-run level of the inflation rate, because the yield curve did not remarkably change when the BOJ made the announcement.

equals $\theta_x - \theta_\pi = 0.45\%$, the term spread between them is only 0.68 percent, which is much smaller than the spread for the nominal rates 2.44 percent. That is, the real term structure is much flatter than the nominal one, and the average slope of the nominal term structure is mainly due to the inflation risk premium.

Instead of summarizing the calibrated parameter values for the continuous-time model, we report those for the discrete-time model in Table 5 for facilitating intuitive interpretation. Panel (a) reports the parameter values for the VAR represented by (14), while Panel (b) reports those for the factor process under the risk-neutral measure, the discrete version of (13)[15] :

$$\mathbf{z}_t - \tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\Phi}}(\mathbf{z}_{t-1} - \tilde{\boldsymbol{\theta}}) + \tilde{\boldsymbol{\Sigma}}_\eta \tilde{\boldsymbol{\eta}}_t \,. \tag{16}$$

The volatility parameters $\boldsymbol{\Sigma}_\eta$ and $\tilde{\boldsymbol{\Sigma}}_\eta$ are calculated with the Cholesky decomposition. The impulse response functions under the actual probability measure are displayed in Figure 7. The two elements of the innovation $\boldsymbol{\eta}_t$ are called inflation and shadow rate innovations, respectively. The response of the inflation rate to inflation innovation is very persistent, and does not return to zero even in 120 months. This is highly consistent with Japan's experience, which faced extensive and pervasive deflationary pressures. The strong and persistent response of the shadow rate to an inflation innovation contributes toward preventing the persistent inflation rate from expanding. On the other hand, a

[15] Note that the long-run levels $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ are the same as those for the continuous-time model. The volatility parameters under the actual probability and risk-neutral measures in the discrete-time model, $\boldsymbol{\Sigma}_\eta$ and $\tilde{\boldsymbol{\Sigma}}_\eta$, can be different from one another, even though they are identical in the continuous-time model.

shadow rate innovation raises the shadow rate and suppresses inflation. As Figure 8 shows, negative shadow rate innovations are observed when the BOJ commenced the ZIRP and QMEP, and raised its quantitative money target from 10-15 to 27-32. This implies that the BOJ's policies support lowering the shadow rate and thus the nominal yield curve[16].

Table 6 reports the fit of the macro-finance model to the data. The model fit is just little worse than that for the yield-only model shown in Table 3, although, as we discussed earlier, we do not estimate the parameters using the most efficient methods,.

5.3. Real Term Structure

The real term structure can be obtained using the following equation:

$$\exp\{-n \cdot r_t^{(T)}(\mathbf{z}_t)\} = \tilde{E}_t\left[\exp\left\{-\int_t^{t+T} r_s ds\right\}\right], \tag{17}$$

where $r_t^{(T)}(\mathbf{z}_t)$ is the T-year real rate. Here r_t is the spot real rate defined as the spot nominal rate i_t minus the spot inflation rate. Since the spot inflation rate is unobservable, we use the year-on-year percent change π_t as the proxy.[17] The spot real

[16] The shadow rate innovation is partially contaminated with influences other than monetary policy. For example, the largest innovation in December 1998 is a famous case caused by a statement by a former Finance Minister against continuing the purchase of government bonds by a government -related agency.

[17] Conceptually, shorter-run inflation, such as month-on-month percentage changes, may be better for the proxy of the spot inflation rate. However, the shorter-run inflation rates are too noisy, and are subject to a large measurement error, which may cause a significant bias in our exercise. On the other hand, our approach can be criticized in the timing problem, in which the year-on-year rate lags the spot rate by around six months on average. We agree that this problem may be crucial for quantifying the real rates, particularly for shorter maturities. But for longer maturity—which interests us the

rate can then be represented as[18]

$$
\begin{aligned}
r_t &= i_t - \pi_t \\
&= \begin{cases} x_t - \pi_t & \text{if } x_t > 0 \\ -\pi_t & \text{otherwise}. \end{cases}
\end{aligned}
\tag{18}
$$

Equation (18) implies an interesting property of the real rate under the zero interest rate. To aid our intuition, let us conduct a static analysis. Suppose the shadow rate follows a very simple linear rule:

$$
x_t = -1.67\% + 3.11\pi_t . \tag{19}
$$

The coefficients are obtained by the OLS regression using the mean-adjusted shadow rate and inflation rate. In fact, as the impulse response functions seen in Figure 7, the response of the shadow rate to an inflation innovation is slower than implied by equation (19), since the dynamics of the shadow rate depends on its lag. Thus equation (19) can be interpreted as the average relationship just for the static analysis.[19] According to equation (19), the shadow rate is negative when the inflation rate is lower than 0.54 percent. Thus equation (18) can be rewritten as

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most—this problem should be less harmful.

[18] We assume there is no inflation risk premium for the spot rate, following Ang and Beckert (2005).

[19] The coefficient on the inflation rate in equation (19), 3.11, may be regarded to be too large in comparison with typical values such as 1.5 in Taylor's (1993) rule for U.S. data. One plausible interpretation is that if monetary policy is likely to be less effective in Japan than in the United States, the BOJ may have to respond to inflation more aggressively. This seems to be true even out of our sample. For example, in the bubble economy around 1990, inflation peaked at 3.2 percent in December 1990, while the call rate peaked at 8.3 percent in March 1991, as shown in Figure 1. These numbers are fairly consistent with (19).

$$r_t = \begin{cases} -1.67\% + 2.11\pi_t & \text{if } \pi_t \geq 0.54\% \\ -\pi_t & \text{otherwise} \end{cases}. \tag{20}$$

This means that the spot real rate can be represented with an increasing function of the inflation rate when the inflation rate is higher than 0.54 percent, and a decreasing function otherwise, as illustrated in Figure 9. The spot real rate reaches the lowest, -0.54 percent, when the inflation rate is 0.54 percent. This figure also confirms that the real and nominal spot rates reach long-run levels of 0.45 percent and 1.45 percent respectively, when the inflation rate hits the long-run level of 1 percent.

Now let us move from the static analysis to the results from our dynamic model. Figure 10 displays the model-implied steady-state nominal and real term structures, which are obtained as the yield curves when the factors are equal to the long-run levels as $x_t = \theta_x$ and $\pi_t = \theta_\pi$. This figure shows that the term structure of real interest rates is very flat around the long-run level 1.14 percent. Thus the term structure of inflation compensations, defined as the nominal rates minus the real rates, is sharply upward-sloping, which suggests a failure of the Fisher hypothesis[20]. Interestingly, this result is consistent with empirical studies for U.S. data such as those from Ang and Beckert (2005) and Buraschi and Jiltsov (2005). For example, Ang and Beckert report that the unconditional average of term structure has a fairly flat shape, around 1.44

---

[20] The difference between the nominal and real rates is called inflation compensation, or the breakeven inflation rate. The Fisher hypothesis argues that the inflation compensation is equal to the expected inflation *for any maturity*. Thus, this hypothesis implies that the inflation compensation at steady state for any maturity equals the steady state inflation rate, and so the difference between the nominal and real yield curves should be flat. Note that inflation compensation is not equal to the expected inflation rate by an inflation risk premium, if the Fischer hypothesis fails.

percent,, which is only slightly higher than our result for Japan's data.

Figure 11 shows the time-variation of real interest rates. Since July 1997, for most of our sample, the shadow rate has been negative, and thus the spot real rate exactly equals the deflation rate, in accordance with (18). Therefore the spot real rate in Panel (a) is negatively correlated with the inflation rate. The longer-term real rates have moved more smoothly. An important reason is that the expected inflation rates for longer maturities vary more gradually as displayed in Figure 12. Note that in January 2006, when the inflation rate was 0.5 percent, the real rates reached the minimum since 1998, when the Japanese economy had already in effect been trapped in deflation. This is consistent with Figure 9, in which the spot real rate reaches the bottom when inflation is around 0.5 percent. Panel (b) of Figure 11 displays the selected real yield curves and confirms that the real yields are fairly stable except for short-maturities. This implies that a large part of the time-variation of the nominal rates is attributable to that of the inflation compensations. The stable real rates are also confirmed in U.S. data by Buraschi and Jiltsov (2005), and by Ang and Beckert (2005).

Let us summarize the results in terms of the monetary policy effect on the yield curve. The BOJ successfully lowered the nominal yield curve, mainly by the policy duration effect, but failed to lower real rates, especially in 2001-2003, when the Japanese economy faced the most serious deflationary pressure. As suggested by the static analysis and Figures 11 and 12, the expected inflation rates are a very important factor for real interest rates under the zero interest. Since the expected inflation rates are harder to control than the expected nominal short rates are, the BOJ faced a difficulty in lowering the real yield curve.

6. Conclusion

The Japanese economy faced serious deflation and the zero bound of the short-term interest rate, which made traditional monetary policy tools less effective. This unique situation has prevented us from applying the recently developed macro-finance approach, the literature of which has provided a number of helpful insights on the relationship between macroeconomics and financial markets, particularly in the United States. Our study overcomes this challenge by combining the macro-finance approach with Black's model of interest rate as options. Using this combined model, we investigate the estimated shadow rate and the model-implied term structures of nominal and real interest rates to asses the BOJ's policy effect on yield curves.

The main findings are summarized as follows. First, we show that the shadow rate fell into negative territory in 1995, and has been negative since 1997, even when the call rate was 0.5 percent and before the BOJ started the ZIRP. This result suggests that 0.5 percent has been regarded as indistinguishable from 0.0 percent by bond market participants, possibly because the BOJ was considered to have no room to lower the call rate. This finding implies that many studies in the literature using the call rate after 1995 should be viewed with a skeptical eye.

Second, the BOJ successfully lowered the nominal yield curve, mainly by the policy duration effect, but it was hard to control real rates. This is because expected inflation rates are an important factor for real interest rates under zero interest, and harder to control than the expected nominal short rates are.

Third, the real term structure is flatter and more stable than the nominal one. Thus the inflation compensation is sharply upward-sloping, and is an important source of the time-variation of the nominal rate. This result is very consistent with the literature that studies the yield curve in United States, and suggests a failure of the Fisher hypothesis.

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Table 1: Estimation Results for the Yield-only Model

κ	θ	σ	λ_0	λ_1	$\tilde{\theta}$
0.215* (0.025)	0.0145* (0.0038)	0.0168* (0.0009)	-0.318* (0.025)	0.186 (0.887)	0.0389* (0.0019)

Note: Standard errors are in parentheses. The estimates with * are significantly different from zero at the 1 percent level.

Table 2: Comparison of Time Series Properties of the Shadow Rate and Call Rate

	μ	ϕ	σ_η
Shadow rate 1995-2006	0.0003* (0.0000)	0.9822* (0.0020)	0.0048* (0.0002)
Call rate 1980-1994	0.0005 (0.0009)	0.9853* (0.0144)	0.0043* (0.0003)

Note: Standard errors are in parentheses. The estimates with * are significantly different from zero at the 1 percent level. The upper estimates are the implied values using the estimates for the yield-only model, and the standard errors are calculated with the delta method. The lower estimates are obtained by an OLS regression of AR(1) model for the call rate over the sample from January 1980 to December 1994.

Table 3: Fit of the Yield-only Model

Spot	0.5Y	2Y	5Y	10Y
0.25	0.18	0.10	0.30	0.46

Note: Standard deviations of the differences between the observed and model-implied yields (percent per year) are reported.

Table 4: OLS Results Using the Shadow Rate

	Dependent variable	Sample	Lag	Inflation rate	R^2
(1)	Shadow rate	Jun. 1995 -Mar. 2006	- -	2.39* (0.41)	0.32
(2)	Call rate	Jun. 1995 -Mar. 2006	- -	0.33* (0.09)	0.17
(3)	Call rate	Jun. 1995 -Jan. 1999	- -	-0.01 (0.20)	0.00
(4)	Shadow rate	Jun. 1995 -Mar. 2006	0.93* (0.02)	0.29* (0.08)	0.97
(5)	Call rate	Jun. 1995 -Mar. 2006	0.85* (0.03)	0.04* (0.01)	0.94

Note: The results of OLS regressions of the shadow rate or call rate on a constant and the year-on-year percent increase of CPI are reported in (1)-(3). For regressions (4) and (5), a lag of the dependent variable is added as a regressor. Standard errors are in parenthesis and estimated by the Newey-West estimator. The estimates with * correspond to the coefficients that are significantly different from zero at the 1 percent level.

Table 5: Calibrated Parameters of the Macro-finance Model

(a) Parameters for the VAR(1) process under actual probability measure

	$\boldsymbol{\theta}$	$\boldsymbol{\Phi}$		$\boldsymbol{\Sigma}_{\eta}$	
π_t	0.0100	1.0137	-0.0064	0.0014	0.0000
x_t	0.0145	0.2670	0.9254	0.0003	0.0032

(b) Parameters for the VAR(1) process under risk-neutral measure

	$\tilde{\boldsymbol{\theta}}$	$\tilde{\boldsymbol{\Phi}}$		$\tilde{\boldsymbol{\Sigma}}_{\eta}$	
π_t	0.0275	0.8639	0.1230	0.0013	0.0000
x_t	0.0389	0.7690	0.1714	0.0011	0.0015

Table 6: Fit of the Macro-finance Model

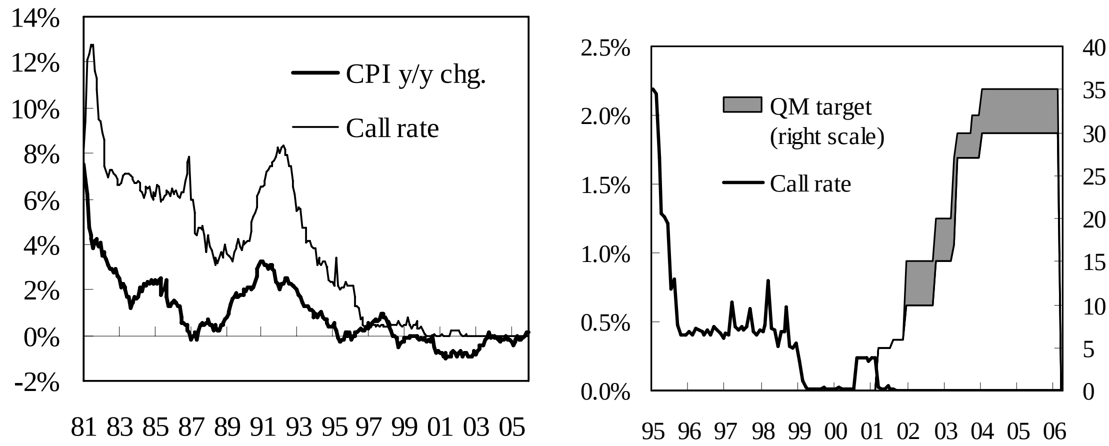
Spot	0.5Y	2Y	5Y	10Y
0.25	0.18	0.22	0.35	0.48

Note: The standard deviations of the differences between the observed and model-implied yields (percent per year) are reported.

Figure 1: Inflation and the Bank of Japan's Polices

(a) Inflation rate and call rate

(b) QM target and call rate

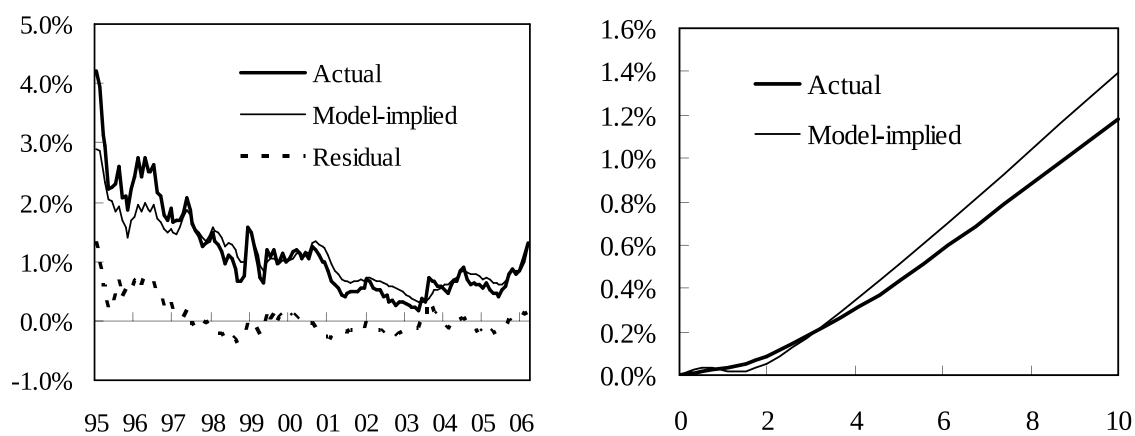


Note: Panel (a) shows the year-on-year change of CPI (excluding fresh food) and the call rate. The CPI is adjusted by the change of consumption tax in 1989 and 1997. Panel (b) reports the call rate and the quantitative money (QM) target (trill. yen) announced by the Bank of Japan.

Figure 2: Fit of the Yield-only Model

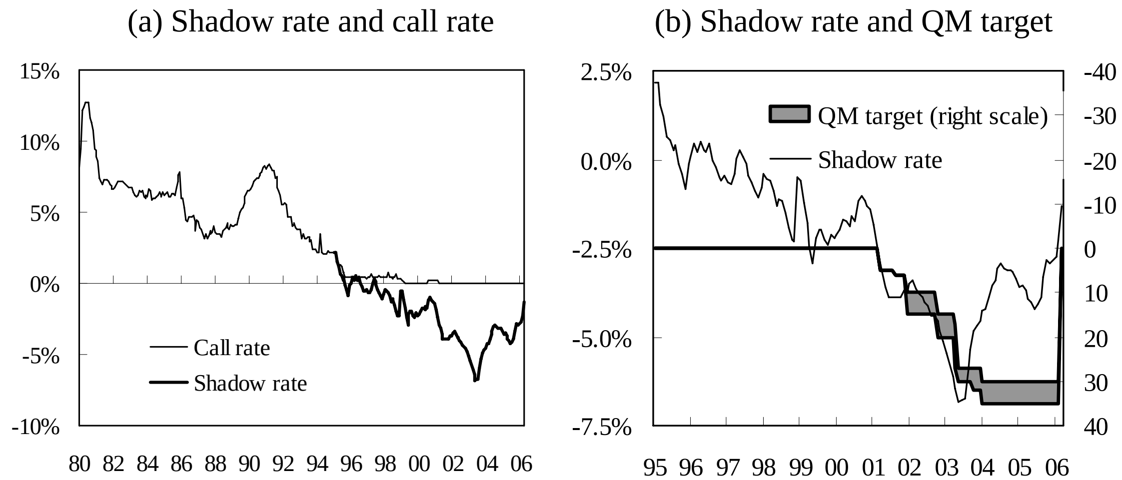
(a) Model fit of the 5-year yield

(b) Average yield curve for 2002-03



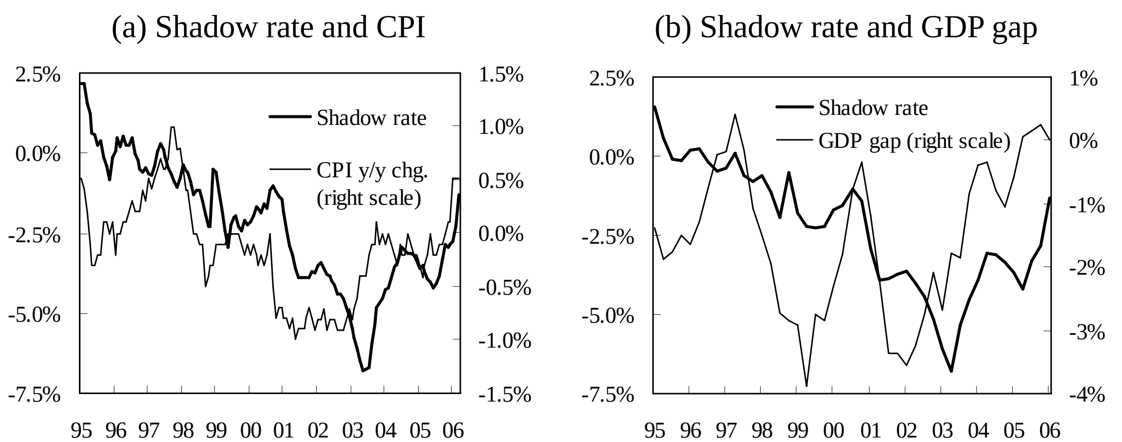
Note: Panel (a) shows the actual and model-implied 5 year maturity rates, and the differences between them. Panel (b) reports the averages of the actual and model-implied yield curves for 2002-03.

Figure 3: Shadow Rate and Monetary Policy Instruments



Note: In Panel (b), the gray area enclosed in fat lines corresponds to the quantitative monetary target of the Bank of Japan (right and reverse scale, trillions of yen).

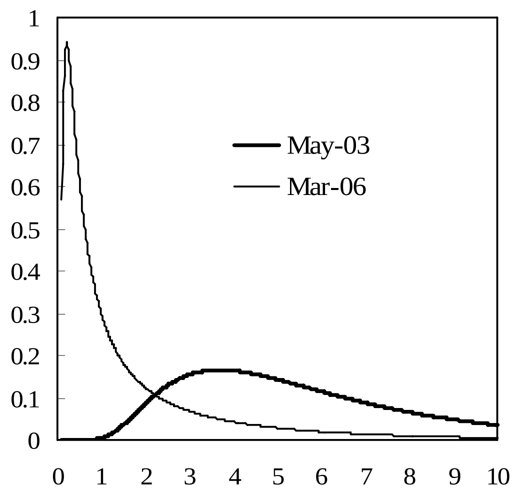
Figure 4: Shadow Rate and Macroeconomic Variables



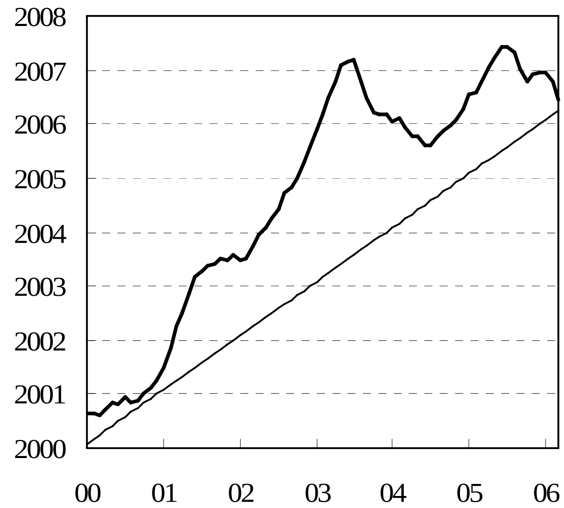
Note: The left panel reports the estimated shadow rate and the year-on-year percent increase of CPI (excluding fresh food, the effect of consumption tax change adjusted) in monthly frequency. The right panel reports the shadow rate and the GDP gap estimated by the Bank of Japan in quarterly frequency.

Figure 5: Predicted Duration of the Zero Interest Rate Policy

(a) Probability density functions

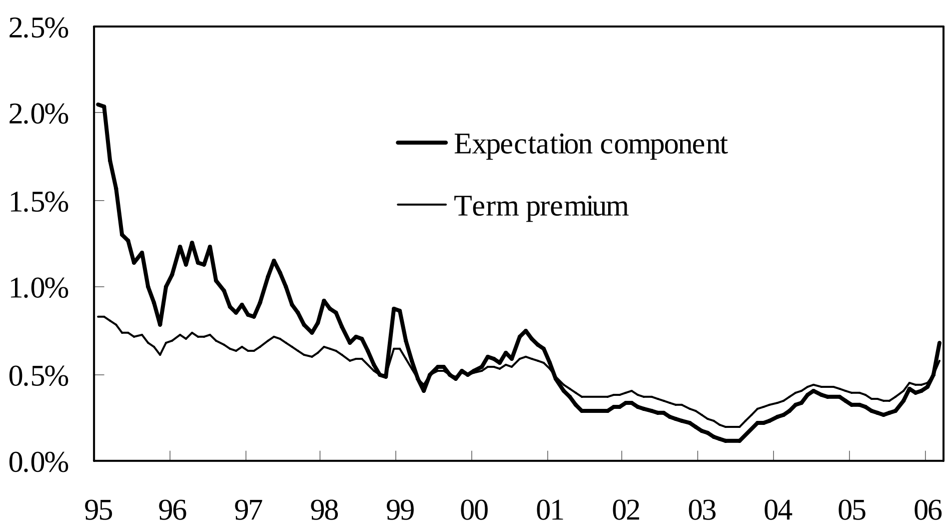


(b) Mode of expected end of ZIRP



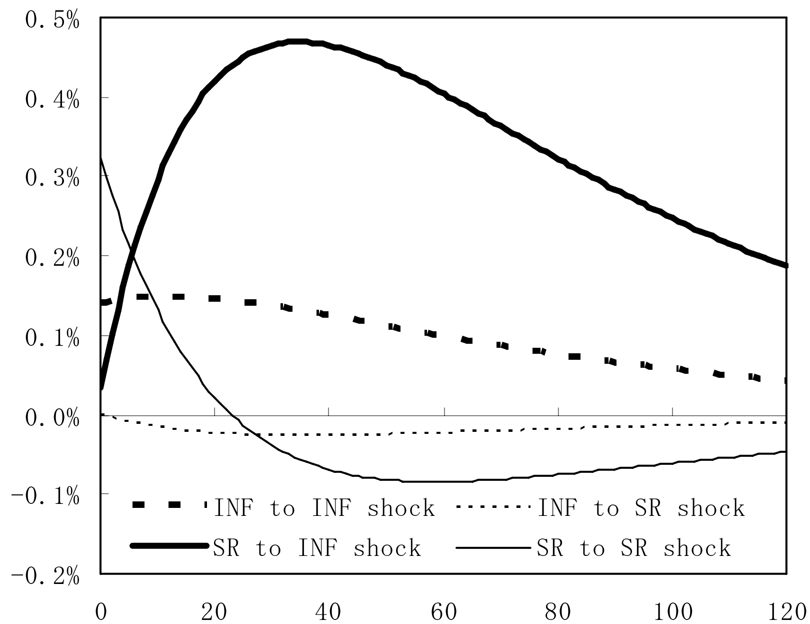
Note: Panel (a) reports the probability density functions of the first hitting-time (years) to the zero shadow rate at March 2003 and March 2006. The x-axis corresponds to the first hitting-time and the y-axis corresponds to the probability. Panel (b) reports the mode of the first hitting-time. The x-axis corresponds to the current time and the y-axis corresponds to the mode of the first hitting-time.

Figure 6: Decomposition of the 5-year Maturity Rate to the Expectation Component and Term Premium



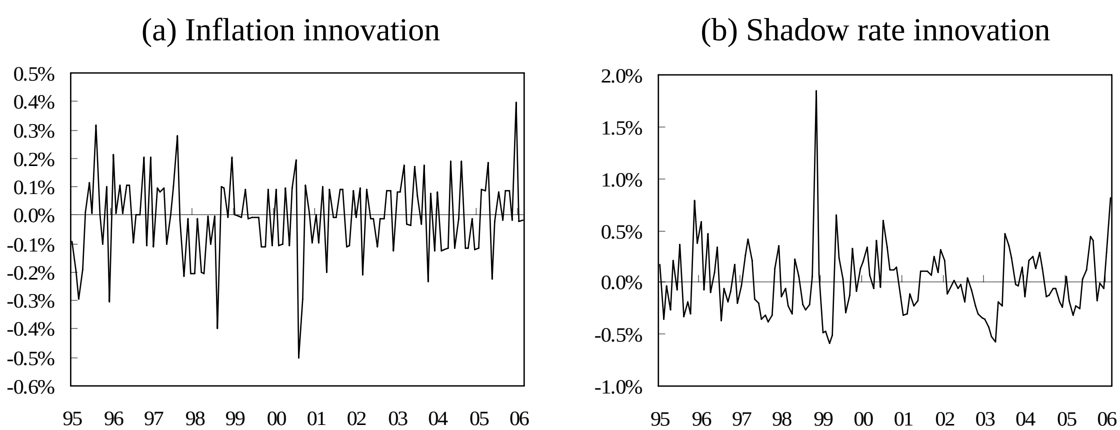
Note: The term premium is calculated using equation (9). The expectation component is obtained as the difference between the estimated 5-year maturity rate and the term premium.

Figure 7: Impulse Response Functions for the Macro-finance Model



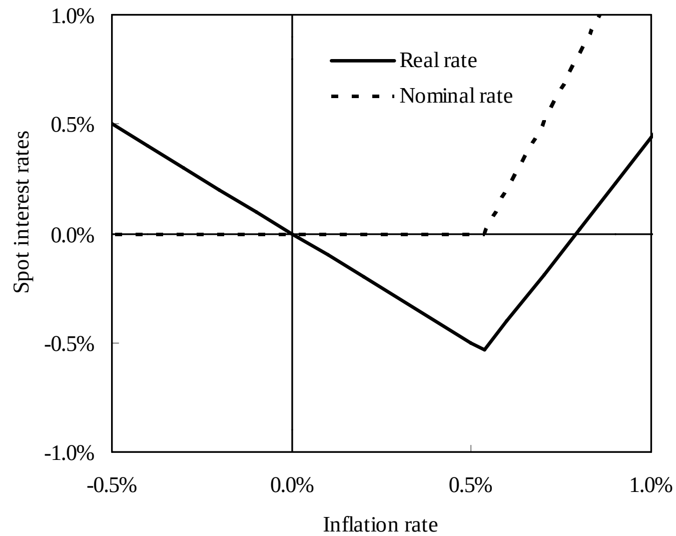
Note: The impulse response functions to one-standard deviation exogenous shocks, obtained from VAR of inflation rate (INF) and the shadow rate (SR) represented in equations (14), are reported. The horizontal axes correspond to horizons (months).

Figure 8: Innovations in the Macro-finance Model



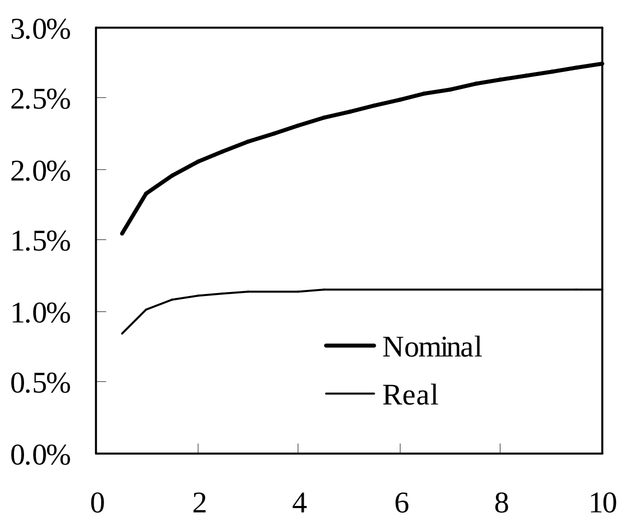
Note: The orthogonalized residuals of the VAR represented in equations (14) are reported.

Figure 9: Illustration of the Relationship Between Interest Rates and Inflation



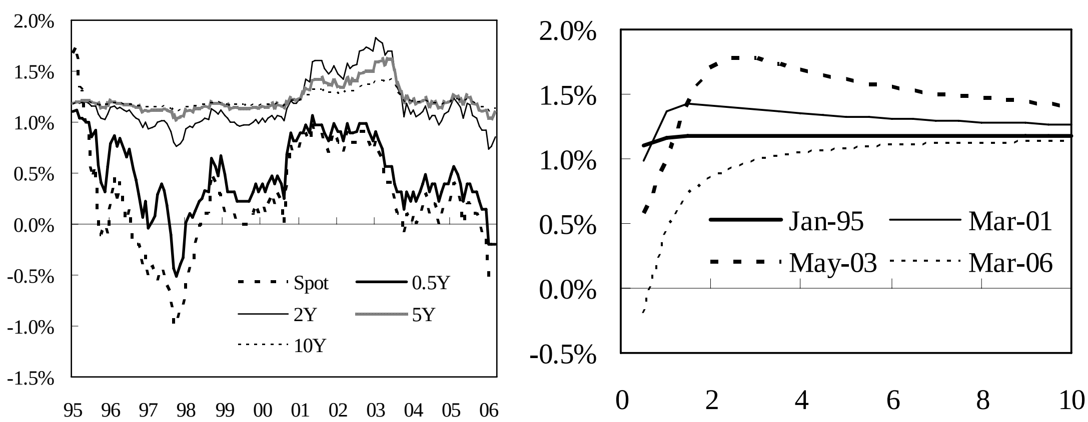
Note: The relationship between the spot interest rates and the inflation rate when the shadow rate obeys a simple linear rule (19) is reported.

Figure 10: Steady States of Nominal and Real Interest Yield Curves



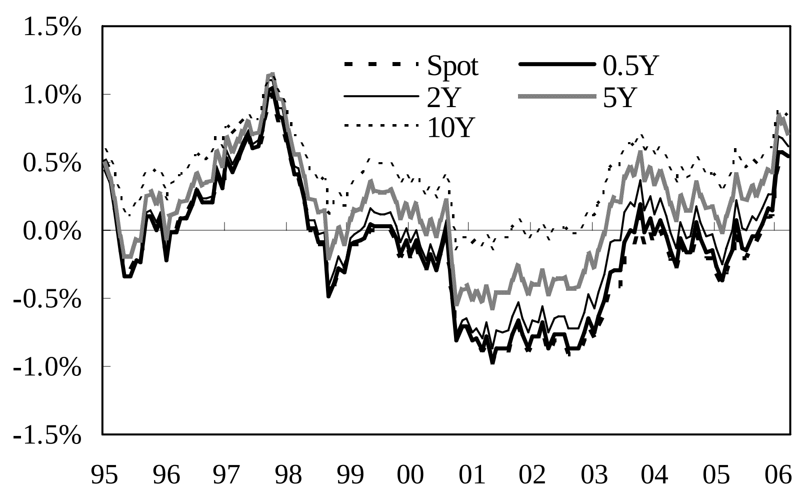
Note: The term structures of nominal and real interest rates when the inflation rate and the shadow rate are the long-run levels are reported.

Figure 11: Real Interest Rates



Note: Panel (a) reports the estimated real rates. Panel (b) reports the model-implied real term structure on selected months.

Figure 12: Expected Inflation Rates



Note: The expected inflation rates implied by the VAR, represented by equation (14), are reported.