Bank of Japan Working Paper Series

Risk Aggregation by a Copula with a Stressed Condition

Toshinao Yoshiba*
toshinao.yoshiba@boj.or.jp

No.13-E-12
September 2013

Bank of Japan
2-1-1 Nihonbashi-Hongokucho, Chuo-ku, Tokyo 103-0021, Japan

* Financial System and Bank Examination Department

Risk Aggregation by a Copula with a Stressed Condition*

Toshinao Yoshiba†

September 2013

Abstract

This paper examines the marginal distributions of stocks and bonds, and a copula between the movement of stock prices and interest rates. Because some widely used aggregation methods such as variance–covariance tend to underestimate the risk of an aggregated portfolio, a copula is utilized for risk aggregation, which captures various dependencies in the median and the tail of marginal distributions, unlike a linear correlation. In this study, various types of copula, including one that simultaneously captures both positive and negative linear correlations, are analyzed under several time periods. We examine data related to the Euro crisis and the post-bubble period in Japan. Our analyses show that widely used risk aggregation methods may overestimate the diversification effect.

Keywords: copula; multivariate distribution; tail dependency; risk aggregation; economic capital

* The author is grateful to Professor Hideatsu Tsukahara (Seijo University), Mr. Masahiko Nishida (Mizuho-DL Financial Technology), and other participants in the "Advanced Risk Measurement" workshop held at the Bank of Japan on February 28, 2013 for their useful comments. Any errors are the author's own. The views expressed here are those of the author and do not necessarily reflect the official views of the Bank of Japan or its Financial System and Bank Examination Department.
† Financial System and Bank Examination Department (toshinao.yoshiba@boj.or.jp)

1. Introduction

Various types of risk aggregation with a diversification effect are applied in financial risk management. Some global banks calculate firm-wide economic capital by aggregating credit, market, operational, and other risk categories with a diversification effect. In contrast, most Japanese regional banks aggregate credit and market risk categories without the diversification effect. They usually aggregate the risk of bonds and equities with the diversification effect in their market portfolio by quantifying a linear correlation between stock prices and interest rates.

Recent Japanese market data have shown a positive linear correlation between stock prices and interest rates, which implies an increase in bond prices together with a fall in stock prices. So, the measured risk of value-at-risk (VaR) or expected shortfall (ES) for the aggregated bond and equity portfolio becomes much smaller than the sum of the risk measures for those sub-portfolios. The reduction in the risk measure for an aggregated portfolio is called the diversification effect. Widely used aggregation methods that analyze recent Japanese data can sometimes show up to a 60% diversification effect.

Some widely used aggregation methods tend to underestimate the risk of an aggregated portfolio. For example, variance–covariance (VCV) methods assume a linear correlation between some stock price and interest rate movements, although the correlation varies according to the size and/or direction of such movements. To avoid such problems, a copula is utilized for risk aggregation, which captures various dependencies in the center and the tail of marginal distributions, unlike a linear correlation. The Basel Committee on Banking Supervision [2010] indicates a copula method for risk aggregation, citing some parametric copulas.

A copula can be applied to measure financial risk by considering a stressed condition. First, a bivariate copula with both positive and negative linear correlations can be applied. Using this type of copula, negative correlation can be taken into account to measure the aggregated risk from positively correlated historical data on average. Second, a copula estimated from stressed data can be applied. Even if marginal distributions are estimated with recent historical data, a copula can be estimated separately with long-past data or other market data in stress situations. For example, the Euro crisis data in Spain or Italy or the post-Bubble data in Japan where stock prices and bond prices jointly plunged (while interest rates rose) can be utilized for the copula estimation.

This paper is organized as follows. Section 2 refers to the setting of data analysis. Section 3 estimates a copula by using data from various markets and periods, and analyzes the risk measures and the diversification effect of the portfolio. Section 4 provides the conclusions and refers to open problems.

2. Setting

2.1. Risk factors and holding period under an unconditional approach

To measure VaR or ES for an aggregated portfolio of equities and bonds, we select two risk factors; daily log return of the stock price and daily changes in the 5-year government interest rate.[1] For Japanese market data, we adopt the Nikkei-225 index as the stock price.

The holding period is set to one day, which is the same as the frequency of the observed returns data. Daily returns are assumed to be independently identically distributed (i.i.d.) under an unconditional approach without using conditional information at the evaluation date.[2]

2.2. Marginal distributions of risk factors

Marginal distributions of risk factors that are adopted in widely used VaR calculation methods have several problems. Gaussian distribution, which is adopted by the VCV method, cannot capture the skewness and kurtosis observed in the historical data. Empirical distribution, which is adopted in the historical simulation (HS) method, cannot capture losses larger than the maximum loss given in the historical data.

Against that background, we adopt the skew-*t* distribution proposed in Azzalini and Capitanio [2003] for each marginal distribution.[3] The distribution has four parameters:

[1] We use generic interest rates calculated by Bloomberg for Japanese, Spanish, and Italian 5-year government interest rates.
[2] For the difference in unconditional and conditional approaches, see Miura and Oue [2000], McNeil, Frey, and Embrechts [2005], and Isogai [2013].
[3] Kubota [2009] adopts double exponential (Laplace) distribution to capture kurtosis. Miura and Oue [2000] applies logistic distribution to the daily returns of the USD/JPY exchange rate. Isogai

location μ, scale σ, degree of freedom ν, and skewness λ. The density is given as

$$f(x) = \frac{2}{\sigma} t_{1,\nu}\left(\frac{x-\mu}{\sigma}\right) T_{1,\nu+1}\left(\lambda \frac{x-\mu}{\sigma}\sqrt{\frac{\nu+1}{(x-\mu)^2/\sigma^2+\nu}}\right), \tag{1}$$

where $t_{1,\nu}(\cdot)$ is the density function of Student's *t*-distribution with the degree of freedom ν and $T_{1,\nu+1}(\cdot)$ is the cumulative distribution function of Student's *t*-distribution with the degree of freedom $\nu+1$.[4]

Table 1 shows the maximum likelihood estimation for the marginal skew-*t* distributions estimated using 5-year historical data from October 1, 2007 to October 1, 2012. The degree of freedom parameter ν is 3.6 for the stock price and 2.9 for the interest rate, which indicates a much fatter tail than Gaussian distribution.[5]

Table 1. Estimated parameters and the 99 percentile point

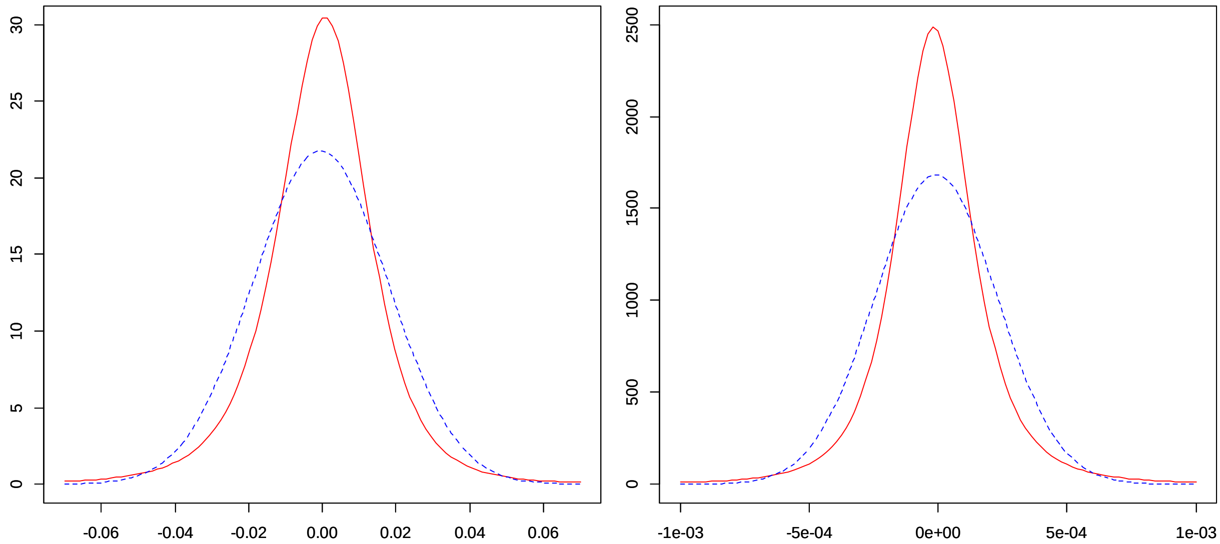
	location (μ)	scale (σ)	shape (λ)	d.f. (ν)	99 percentile
Stock price	0.002832	0.012462	–0.267	3.625	–0.05215
Interest rate	–0.000030	0.000148	0.129	2.900	0.00071

Figure 1 depicts the estimated density, which is much different from Gaussian density (the blue dotted curve in Figure 1). Figure 2 depicts QQ plot and goodness-of-fit tests for the estimated skew-*t* distributions. Neither test rejects the null hypothesis even at the 10% confidence level. The skew-*t* distribution is accepted as each marginal distribution.

[2013] adopts truncated stable distribution for the daily return of the Nikkei-225 index. Although the truncated stable distribution can be applied to the analyses in this paper, estimation of the distribution is difficult and accompanied by arbitrariness in setting truncation. So, we adopt the skew-*t* distribution for marginal distribution.

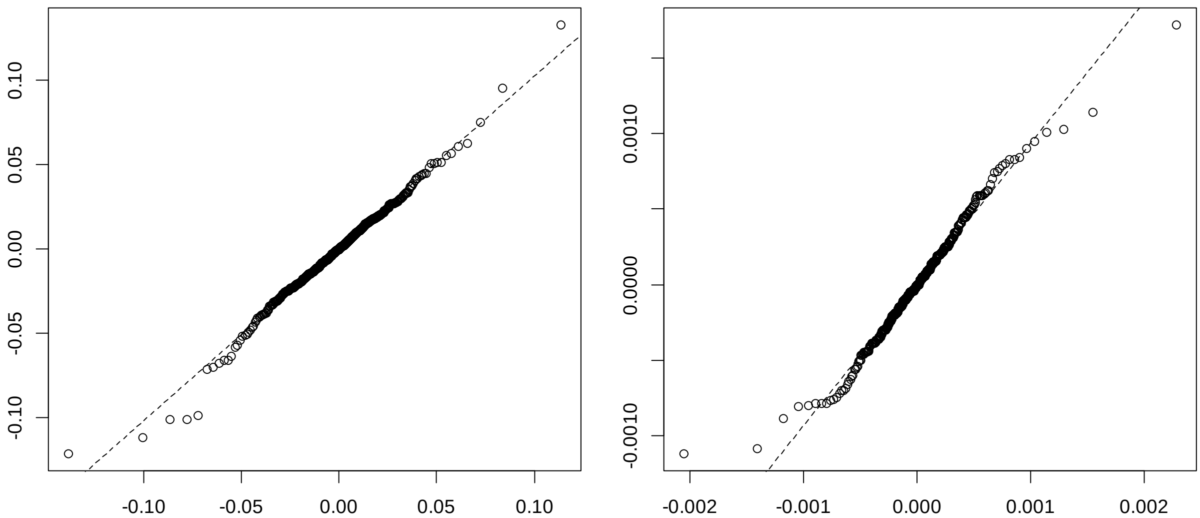
4 The skew-*t* distribution is reduced to Student's *t* distribution with skewness parameter λ=0. Furthermore, Student's *t* distribution converges to Gaussian distribution as $\nu \to \infty$. In financial econometrics, other types of skew-*t* distribution, such as Hansen [1994] or Fernández and Steel [1998] are used in some cases. See Aas and Haff [2006] for variation of skew-*t* distribution.

5 If the degree of freedom parameter ν is less than two, the variance becomes infinite. If ν is less than four, the fourth moment (kurtosis) becomes infinite.



Note: The blue dotted curve is fitted to Gaussian distribution.

Figure 1. Estimated density (left: stock price, right: interest rate)



Stock price return		Interest rate movement	
Kolmogorov–Smirnov	Anderson–Darling	Kolmogorov–Smirnov	Anderson–Darling
D = 0.016 *p* value = 0.900	AD = 0.307 *p* value = 0.933	D = 0.032 *p* value = 0.161	AD = 0.664 *p* value = 0.590

Notes: In the upper QQ plot, the horizontal axis denotes the theoretical value of skew-*t* distribution and the vertical axis denotes data value. In the lower test for goodness of fit, null hypothesis H_0 is "Data have skew-*t* distribution." If the *p* value is low, the null hypothesis H_0 is rejected.

Figure 2. QQ plot and goodness-of-fit test (left: stock price, right: interest rate)

2.3. Correlation structure between risk factors

2.3.1. Copula

We capture the correlation structure between risk factors by using a copula. A copula represents a joint distribution as a function of the marginal distributions. Using a copula $C(u_1,\ldots,u_d)$, the joint cumulative probability (distribution function) of the d random variables is represented as

$$\Pr(X_1 \le x_1,\ldots,X_d \le x_d) = C(\Pr(X_1 \le x_1),\cdots,\Pr(X_d \le x_d)). \qquad (2)$$

To employ the maximum likelihood method, the copula density $c(u_1,\ldots,u_d)$ for the copula $C(u_1,\ldots,u_d)$ is defined as[6]

$$c(u_1,\ldots,u_d) = \frac{\partial C(u_1,\ldots,u_d)}{\partial u_1 \cdots \partial u_d}. \qquad (3)$$

There are two approaches to estimate a copula from a pseudo sample (u_{11},u_{21}), …, (u_{1N},u_{2N}).[7] The first is a non-parametric approach based on joint empirical distribution. The second is a parametric approach to assume a parametric copula and estimate the parameters.

A nonparametric copula $C_{NP}(u_1,u_2)$ is constructed from a joint empirical distribution function of the pseudo sample (u_{11},u_{21}), …, (u_{1N},u_{2N}).[8] The copula is constructed as cumulative joint probabilities less than or equal to (u_1,u_2) after giving $1/N$ probability weight to each pair (u_{11},u_{21}), …, (u_{1N},u_{2N}) in $[0,1]\times[0,1]$. The histogram for the pseudo sample makes the nonparametric copula density $c_{NP}(u_1,u_2)$.

A parametric copula is estimated by maximizing the log likelihood function for the pseudo sample (u_{11},u_{21}), …, (u_{1N},u_{2N}). The log likelihood function consists of the density functions (3).

---

[6] Copula density does not necessarily exist. We assume that copula $C(u_1,\ldots,u_d)$ is continuously differentiable and that the copula density exists.

[7] Let $x_{11},\ldots,x_{1N}$ be the stock return data, and $x_{21},\ldots,x_{2N}$ be the interest rate movement data. Using the distribution functions $F_1(\cdot)$ and $F_2(\cdot)$ estimated from the stock return and interest rate movement data respectively, the pseudo sample (u_{11},u_{21}), … , (u_{1N},u_{2N}) is obtained by $u_{ij} = F_i(x_{ij})$ for $i=1,2;\ j=1,\ldots,N$.

[8] The empirical copula, a representative nonparametric copula, is defined by the empirical marginal distributions. The nonparametric copula in this paper assumes skew-*t* distribution for each margin, and differs from the empirical copula.

Tail dependence is one of the key properties for a parametric copula. The lower-tail dependence λ_L between two variables is defined as

$$\lambda_L = \lim_{u \to 0} \Pr(U_2 \le u \mid U_1 \le u) = \lim_{u \to 0} \frac{C(u,u)}{u}. \tag{4}$$

The upper-tail dependence λ_U is defined as

$$\lambda_U = \lim_{u \to 1} \Pr(U_2 > u \mid U_1 > u) = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u}. \tag{5}$$

Thus, λ_L and λ_U represent the strength of lower and upper tail dependences, respectively.

2.3.2. Parametric copulas

Six parametric copulas are utilized in the following sections. Three of them are Archimedean copulas, which include the Gumbel, Clayton, and Frank copulas. The other three are implied copulas, which include the Gaussian, (Student's) *t*, and mixed-Gaussian copula.

Table 2 summarizes the expression and the tail dependence of the three Archimedean copulas. The three copulas have different tail dependence. While the Gumbel copula has upper-tail dependence and the Clayton copula has lower-tail dependence, the Frank copula has neither tail dependence nor symmetric dependence.

Table 2. Bivariate Archimedean copulas and their tail dependence

Copula	Parameter	Expression	Tail dependence	
			Upper λ_U	Lower λ_L
Gumbel	γ	$\exp\{-((-\ln u_1)^{\gamma} + (-\ln u_2)^{\gamma})^{1/\gamma}\}$	$2 - 2^{1/\gamma}$	0
Clayton	α	$(u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha}$	0	$2^{-1/\alpha}$
Frank	δ	$-\frac{1}{\delta}\ln\left(1 + \frac{(e^{-\delta u_1} - 1)(e^{-\delta u_2} - 1)}{(e^{-\delta} - 1)}\right)$	0	0

The Gumbel and Clayton copulas are defined for positively correlated data. For the bivariate case, they can be applied for negatively correlated data by rotating one axis around the other. For example, suppose that the data implies strong dependence between a fall in the first variable (stock price) and a rise in the second variable (interest rate). The Clayton copula with lower-tail dependence can be applied for the data by rotating the second axis around the first (rotating the interest rate axis; Rot. IR). The copula density is defined as $c(u_1,u_2)=c_C(u_1,1-u_2)$ by using the Clayton copula density $c_C(u_1,u_2)$. Similarly, the Gumbel copula with upper-tail dependence can be applied for the data by rotating the first axis around the second (rotating the stock price axis; Rot. SP).

Although the Gumbel copula is upper-tail dependent, it can be applied for lower-tail dependent data with the rotated-Gumbel copula. Similarly, a Clayton copula with lower-tail dependence can be applied for upper-tail dependent data with the rotated-Clayton copula. The rotated copula density $c(u_1,\ldots,u_d)$ for a copula density $\hat{c}(u_1,\ldots,u_d)$ is defined as $c(u_1,\ldots,u_d)=\hat{c}(1-u_1,\ldots,1-u_d)$. A rotated copula is also referred to as a survival copula.

Table 3 summarizes the expression and tail dependence of the three implied copulas; Gaussian, *t*, and mixed-Gaussian copula, each of which adopts a corresponding multivariate distribution as follows. The Gaussian copula is implied in a multivariate Gaussian distribution and the *t* copula is implied in a multivariate Student's *t* distribution. The *t* copula has a parameter ν, which corresponds to the degree of freedom for the original multivariate *t* distribution. The range of the parameter ν is $[3,\infty)$. The *t* copula converges to the Gaussian copula as $\nu\rightarrow\infty$. The parameter ν in *t* copula is interpreted as a tail-dependence parameter.[9] Likewise, a mixed-Gaussian copula is implied in mixed-Gaussian distribution. In the bivariate case, the mixed-Gaussian distribution is mixed with positively correlated Gaussian distribution and negatively correlated Gaussian distribution in the ratio of θ : $1-\theta$. The mixed Gaussian copula will extract negatively correlated Gaussian copula from positively linear correlated data.

[9] This point is confirmed by the fact that $t_{\nu+1}(-\sqrt{(1-\rho)(\nu+1)/(1+\rho)})$ is the largest in $\nu=3$ and becomes smaller as ν increases with a given ρ.

Table 3. Bivariate implied copulas and their tail dependence

Copula	Parameter	Expression	Tail dependence $\lambda_U = \lambda_L$
Gaussian	ρ	$\mathbf{\Phi}(\Phi^{-1}(u_1), \Phi^{-1}(u_2) \mid \rho)$	0
t	$\rho,\ \nu$	$\mathbf{t}_\nu(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2) \mid \rho)$	$2t_{\nu+1}\left(-\sqrt{\frac{(1-\rho)(\nu+1)}{(1+\rho)}}\right)$
Mixed-Gaussian	θ, ρ_1, ρ_2	$\theta\mathbf{\Phi}(\Phi^{-1}(u_1), \Phi^{-1}(u_2) \mid \rho_1) + (1-\theta)\mathbf{\Phi}(\Phi^{-1}(u_1), \Phi^{-1}(u_2) \mid \rho_2)$	0

Notes: $\mathbf{\Phi}(\cdot,\cdot \mid \rho)$ denotes bivariate standard Gaussian distribution function with correlation ρ. $\mathbf{t}_\nu(\cdot,\cdot \mid \rho)$ denotes bivariate t distribution function with degree of freedom ν and correlation ρ. $\Phi^{-1}(\cdot)$ denotes the inverse function of univariate standard Gaussian distribution function. $t_\nu(\cdot)$ denotes univariate t distribution function and $t_\nu^{-1}(\cdot)$ denotes its inverse function.

2.4. Risk measures and diversification effect

We consider a sample portfolio that consists of 700 billion yen for the 5-year discount bond and 50 billion yen for the stock; this is representative of the average portfolio of Japanese regional banks. We adopt daily 99% VaR and 97.5% ES estimates for risk measures.[10]

We calculate 99% VaR and 97.5% ES as follows. Let AS and AB be the amount of stocks and bonds, respectively. Suppose that the stock price changes as $\ln S_t \to \ln S_t + \Delta S$ and the 5-year interest rate changes as $r_t \to r_t + \Delta r$. We calculate the change in the market portfolio value ΔPV as

$$\begin{aligned} \Delta PV &= AS\frac{S_t e^{\Delta S}}{S_t} + AB\frac{e^{-(r_t+\Delta r)T}}{e^{-r_t T}} - (AS + AB) \\ &\cong AS\Delta S - AB(\Delta r)T. \end{aligned} \tag{6}$$

Sample data for $(\Delta S, \Delta r)$ obtained by converting the quantile function for each marginal distribution from simulated data by using the specified copula. We calculate 99% VaR as lower than one percentile for the right-hand side of the equation (6) with an opposite sign. Similarly, 97.5% ES is calculated as the average up to lower 2.5 percentile for the right-hand side of equation (6) with an opposite sign.

The simple sum of 99% VaR for each risk category is given as follows. The 99% VaR for the equities is given by ΔS with the lower one-percent value. The 99% VaR

[10] If the portfolio profit–loss distribution is Gaussian, 97.5% ES almost equals 99% VaR.

for the bonds is given by Δr with the upper one-percent value. The 99 percentile points in Table 1 show the one-percent values. We denote the lower and upper one-percent values as $\Delta S_{1\%}$ and $\Delta r_{99\%}$, respectively. The simple sum of 99% VaR is given as

$$\text{Simple sum of VaR} = -AS(\Delta S_{1\%}) + AB(\Delta r_{99\%})T\ . \tag{7}$$

With our data, the simple sum of 99% VaR is 5.08 billion yen as shown in Table 4. Based on the assumption that VaR satisfies subadditivity,[11] the simple sum is the maximum value for the portfolio VaR.

Similarly, the simple sum of 97.5% ES for each risk category is given by applying the lower 2.5% average $\overline{\Delta S}_{2.5\%}$ for ΔS and the upper 2.5% average $\overline{\Delta r}_{97.5\%}$ for Δr to equation (6). This is expressed as follows:

$$\text{Simple sum of ES} = -AS(\overline{\Delta S}_{2.5\%}) + AB(\overline{\Delta r}_{97.5\%})T\ . \tag{8}$$

With our data, the simple sum of each ES is 5.59 billion yen as shown in Table 4. Since ES satisfies subadditivity, the simple sum is the maximum value for the portfolio ES.

Table 4. VaR and ES for stocks and bonds, and their simple sum

VaR(99%)		ES(97.5%)	
Stocks	Bonds	Stocks	Bonds
2.61	2.47	2.82	2.77
simple sum		simple sum	
5.08		5.59	

(billion yen)

We measure a diversification effect by the reduction rate of the aggregated VaR and ES from the simple sum of VaR and ES.

3. Data Analysis

This section analyzes diversification effects by using copulas. First, we analyze diversification effects by using the copulas estimated from the recent historical data in Japan. Second, we analyze the diversification effects by using the copulas estimated

[11] If some risk measure $\rho(X_1 + X_2)$ for an aggregated portfolio $X_1 + X_2$ of any sub-portfolio X_1 and X_2 always satisfies $\rho(X_1 + X_2) \le \rho(X_1) + \rho(X_2)$, then the risk measure is subadditive. Theoretically, VaR is not subadditive. However, Daníelsson *et al.* [2013] show that VaR is subadditive in most practical situations.

from the recent Euro crisis data and the post-Bubble period in Japan.

3.1. Recent historical data in Japan

3.1.1. Widely used methods of aggregation

The VCV and historical simulation (HS) methods are widely used to aggregate VaR in financial institutions. Table 5 shows the aggregated 99% VaR and 97.5% ES calculated by the two methods. VCV uses a Gaussian copula and HS uses a nonparametric copula for the aggregation.

Table 5. Diversification effects of widely used VaR and ES

	VaR(99%)	diversification effect	ES(97.5%)	diversification effect
simple sum	5.08	—	5.59	—
VCV	2.30	▲55%	2.31	▲59%
HS	3.02	▲41%	3.27	▲42%
Corr. Method	2.87	▲44%	3.16	▲44%

We see a large diversification effect in the VCV method: 55% for VaR and 59% for ES. The diversification effect in HS is smaller than that in VCV: 41% for VaR and 42% for ES.

To investigate such differences, we introduce the correlation method to calculate the aggregated VaR and ES. This method uses the same linear correlation matrix to aggregate each risk as the VCV method.[12] It assumes skew-*t* distribution for the marginal distributions, which is different from the VCV method. By assuming that Ω is the correlation matrix used in the VCV method, the aggregated VaR by the correlation method is given by

$$\text{Corr. method VaR} = \sqrt{(AS\Delta S_{1\%}, -AB(\Delta r_{99\%})T)\Omega(AS\Delta S_{1\%}, -AB(\Delta r_{99\%})T)^{\mathrm{T}}}, \quad (9)$$

where $\Delta S_{1\%}$, $\Delta r_{99\%}$ are 99 percentile points in Table 1. Similarly, the aggregated ES by correlation method is given by

$$\text{Corr. method ES} = \sqrt{(AS\overline{\Delta S}_{2.5\%}, -AB(\overline{\Delta r}_{97.5\%})T)\Omega(AS\overline{\Delta S}_{2.5\%}, -AB(\overline{\Delta r}_{97.5\%})T)^{\mathrm{T}}}, \quad (10)$$

where $\overline{\Delta S}_{2.5\%}$ and $\overline{\Delta r}_{97.5\%}$ are the lower 2.5% average for ΔS and upper 2.5%

[12] The correlation method is widely used for firm-wide risk aggregation for market, credit, and operational risk categories.

average for Δr, respectively. Table 5 indicates the diversification effect in the correlation method, which at 44%, is close to that in the HS method (41–42%).

On comparing the result in the VCV method with that in the correlation method, we see that the former, which assumes Gaussian distribution for the marginal distributions cannot capture fat-tail properties in real risk factors' distributions. That may cause underestimation of the aggregate risk due to overestimation of the diversification effect.

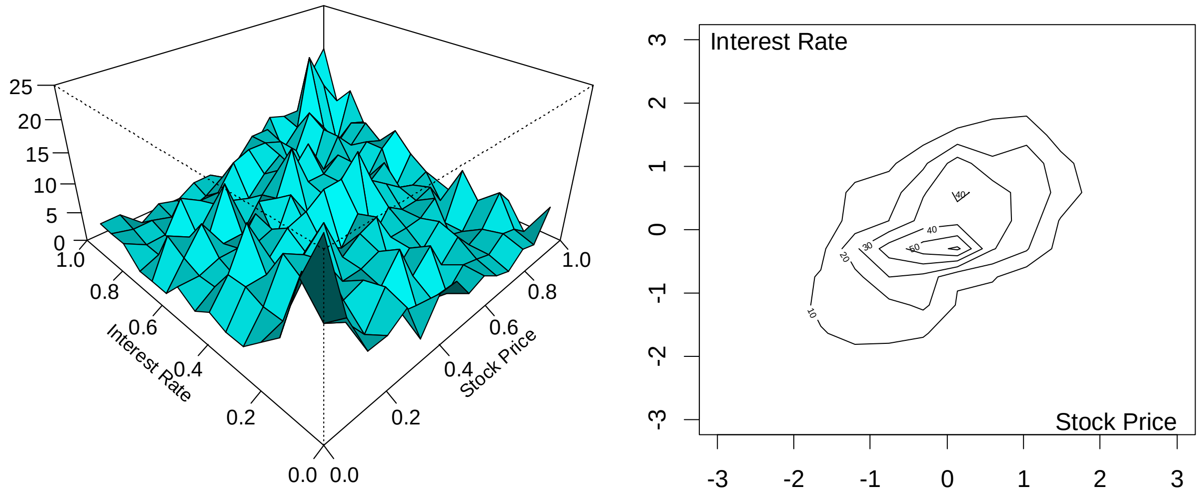
3.1.2. Estimated copula

From here on, we use skew-*t* distribution for the marginal distributions estimated in Table 1. We adopt BIC (Schwarz's Bayesian information criteria) to select the best parametric copula.[13]

This subsection estimates copulas from the recent 5-year daily data in Japan. Figure 3 depicts a nonparametric copula based on the pseudo sample during the period. In Figure 3 (a), the front side with zero for both axes indicates the largest fall in the stock price and interest rate. The figure shows that the frequency is relatively high, which implies that the portfolio loss will be mitigated because bond values rise when stock prices fall. Figure 3 (b) plots the joint density contour after converting marginal cumulative probabilities to the quantiles of standard Gaussian distribution.[14] That contour is diagonally up to the right with an elliptical shape, which implies this pseudo sample have a positive linear correlation.

[13] Some information criteria such as AIC (Akaike information criteria) and BIC are used to select the best copula. Both criteria are calculated by log-likelihood with some penalty about the number of parameters. We adopt BIC for the criteria, which is more penalty about the number of parameters than AIC. BIC is calculated by $-2\ln \ell(\xi) + p \ln N$, where $\ell(\xi)$ is the maximum log-likelihood, p is the number of parameters, and N is the sample size. The model with the lowest BIC is selected.

[14] Contour of the joint density with standard Gaussian margins is a visual representation of the various dependencies in the center and the tail area (see Joe [1997]). If the copula is Gaussian, the contour is elliptical (see Figure 4).



(a) Histogram **(b) Contour of the joint density**

Figure 3. Joint histogram and contour plot for the recent Japanese pseudo sample

Table 6 is the result of the maximum likelihood estimation.[15] A mixed-Gaussian copula is selected by BIC. Within one-parameter copulas, both Frank and rotated-Gumbel have high likelihood (low BIC), while they have quite dissimilar tail dependency. This implies that the dependence structure of the data is too complex to be captured by the one-parameter copula.

Table 6. MLE for the recent Japanese pseudo sample

	Parmeter	Est. Value	Std. Err.	BIC
Gumbel	γ	1.385	0.031	–239.8
Rotated-Gumbel	γ	1.416	0.031	–282.0
Clayton	α	0.662	0.050	–236.8
Rotated-Clayton	α	0.567	0.047	–184.5
Frank	δ	3.188	0.189	–282.6
Gaussian	ρ	0.436	0.021	–251.9
t	ρ	0.466	0.024	–307.2
	ν	5.481	0.918	
Mixed-Gaussian	ρ_1	–0.458	0.124	–313.0
	ρ_2	0.616	0.026	
	θ	0.145	0.036	

Figure 4 depicts a joint density contour for the estimated copula in Table 6 with standard Gaussian margins. To save space, as for the Gumbel copula, the rotated-Gumbel copula is illustrated. As for the Clayton copula, the original Clayton

[15] We do not test the goodness-of-fit of the copulas. See Kojadinovic, Yan, and Holmes [2011] for those tests.

copula is illustrated. Each of them has a higher likelihood (lower BIC) than their rotated copula.

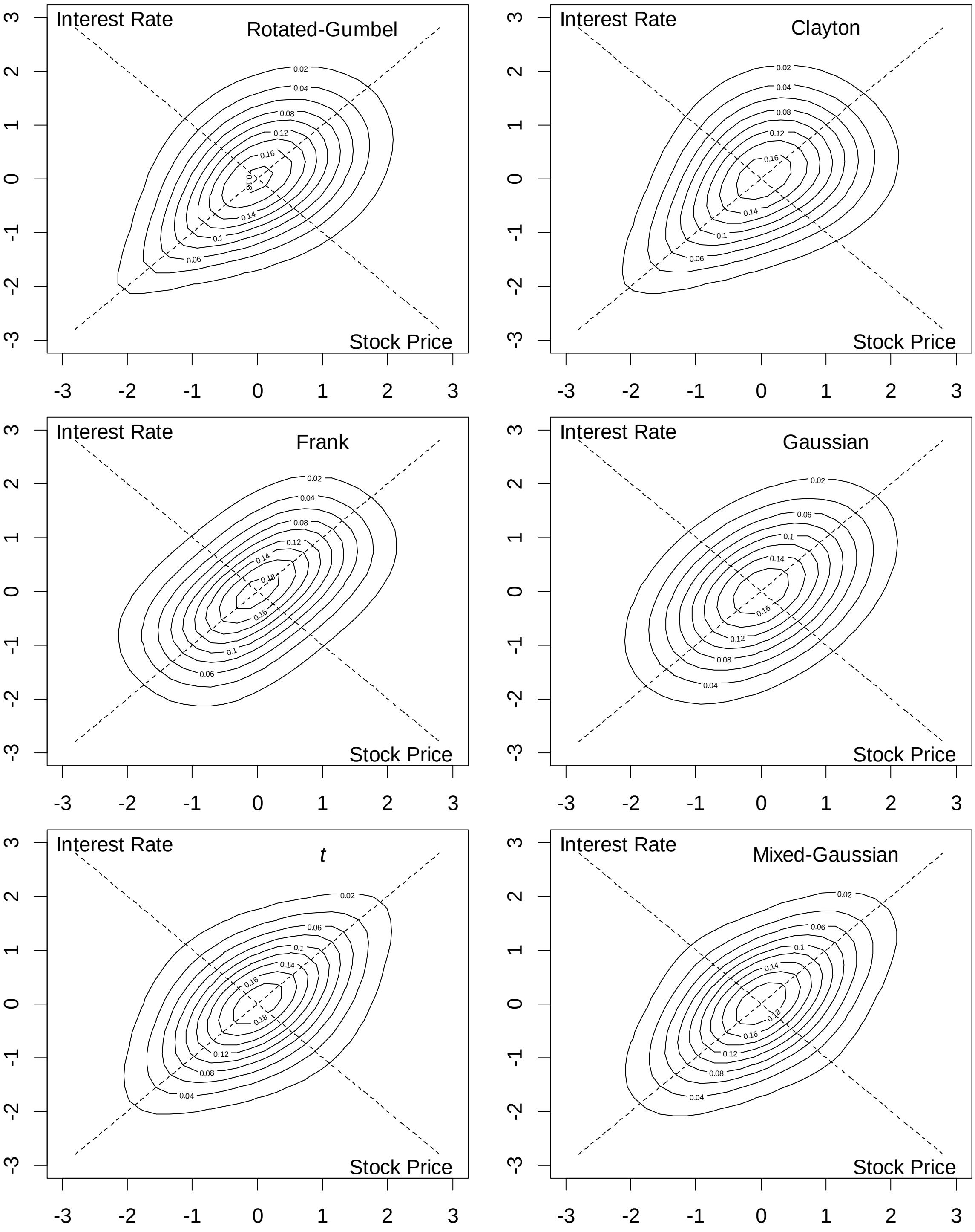


Figure 4. Contour plot of copula density for the recent historical pseudo sample in Japan

3.1.3. VaR and ES for each copula and diversification effect

Table 7 summarizes VaR and ES for the sample portfolio.[16] The joint distribution for risk factors is constructed by each estimated copula with the marginal distributions in Table 1. For the nonparametric copula, the pseudo sample is converted to a set of risk factors by taking quantiles for the marginal skew-*t* distributions. For each parametric copula, we generate 100,000 random bivariate vectors and calculate VaR and ES with the marginal skew-*t* distributions. Iterating the procedure 100 times, we obtain the average and standard deviation of VaR and ES.

Table 7. VaR and ES using copula estimated from recent Japanese data

Copula	VaR(99%)	Std. dev.	diversification effect	ES(97.5%)	Std. dev.	diversification effect
Nonparametric	3.01	––	▲41%	3.24	––	▲42%
Gumbel	2.66	0.03	▲48%	2.90	0.04	▲48%
Rotated-Gumbel	2.58	0.03	▲49%	2.84	0.04	▲49%
Clayton	2.68	0.03	▲47%	2.96	0.04	▲47%
Rotated-Clayton	2.81	0.03	▲45%	3.05	0.04	▲45%
Frank	2.87	0.03	▲43%	3.18	0.04	▲43%
Gaussian	2.65	0.03	▲48%	2.95	0.04	▲47%
t	2.60	0.03	▲49%	2.85	0.04	▲49%
Mixed-Gaussian	4.21	0.04	▲17%	4.55	0.05	▲19%

First, every diversification effect (17–49%) is smaller than that of VCV (55%) in Table 5, which confirms that the Gaussian assumption of VCV for margins causes underestimation of VaR and ES. The diversification effect of the nonparametric copula almost equals that of the HS method in Table 5.[17]

Second, the diversification effect for the *t* copula is larger than that for the Gaussian copula, although the former is more tail dependent than the latter. This result is explained as follows: the lower-tail dependence of the *t* copula (low ν value) increases

[16] The quantiles for each marginal distribution (skew-*t* distribution) are evaluated by 500,000 random numbers. Christoffersen *et al.* [2012] also evaluate quantiles for another skew-*t* distribution with 100,000 random numbers because closed-form (well-approximated) solutions for the quantiles are not known.

[17] Since the marginal distributions for separating the copula and that for combining into a joint distribution are the same for the recent Japanese data, the portfolio loss distribution for the nonparametric copula theoretically coincides with that for the HS method. However, in our analysis, the quantiles for each marginal distribution that is to be combined to a joint distribution are approximated by random numbers as in footnote 16, while the marginal probabilities to separate the copula are calculated more accurately. That difference causes a slight disparity in VaR and ES.

the occurrence of a fall in the stock price and a rise in bond value. That dependence mitigates portfolio loss and results in smaller VaR and ES for the *t* copula than those for the Gaussian copula.

Third, the diversification effect for mixed-Gaussian copula (VaR:17%, ES:19%) selected by BIC is the smallest, and is much smaller than that for the other copulas. Unlike the other parametric copulas, a mixed-Gaussian copula can capture both positive and negative linear correlations so that it fits to the pseudo sample better than the other parametric copulas. The result indicates a negative linear correlation is captured with the frequency of θ = 14.5%; this correlation structure increases the estimate of the portfolio VaR and ES.

3.2. Euro crisis data

This subsection extracts copulas from the recent Euro crisis data in Spain and Italy.[18] Estimating the marginal distributions (stock price return and interest rate movement) by using the recent three years' data from October 1, 2009 to October 1, 2012, we separately estimate the copulas by converting the marginal data to the pseudo sample.

3.2.1. Spain

Adopting IBEX35 as the stock price, Table 8 shows the estimated marginal distributions in separating the copulas.

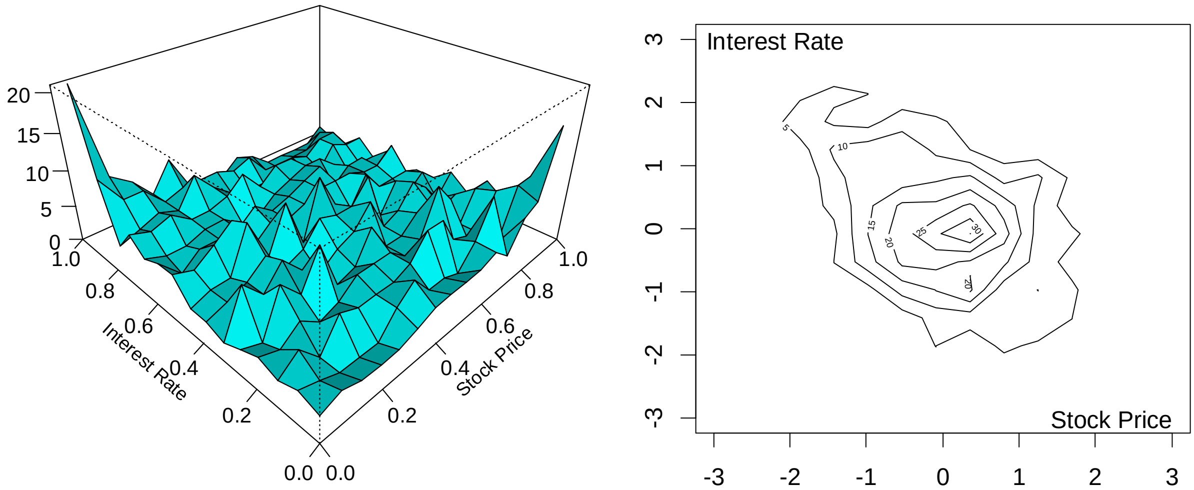
Table 8. Estimated parameter and the 99 percentile point for recent data from Spain

	location (μ)	scale (σ)	shape (λ)	d.f. (ν)	99 percentile
Stock price	–0.000078	0.013825	–0.040	4.792	–0.04829
Interest rate	0.011546	0.069188	–0.076	2.307	0.39611

Figure 5 (a) depicts the histogram for the pseudo sample. Figure 5 (b) depicts the joint density contour with standard Gaussian margins. The pseudo sample is constructed from the returns data by applying each marginal distribution function estimated in Table 8. Unlike Figure 3, the contour is diagonally down to the right with an elliptical shape,

[18] Fukuda, Kan, and Sugihara [2013] analyze the optimal asset composition ratio of stocks and bonds for a bank taking into consideration the correlation between interest rate movements and stock price returns by using the mean-variance approach. They assume a bivariate Gaussian distribution for the risk factors. Our assumption differs regarding those for marginal distribution (skew-*t* distribution) and for various copula.

which implies this data will have a negative linear correlation.



(a) Histogram **(b) Contour of the joint density**

Figure 5. Joint histogram and contour plot for the recent Spanish pseudo sample

Table 9 shows the result for the maximum likelihood estimation of copulas; the *t* copula is selected as the best fit by BIC. Since the pseudo sample has a negative correlation, the copula with rotating one axis (Rot. IR or Rot. SP) is applied for the Gumbel and Clayton copulas as defined in Section 2.3.2.[19] The Gumbel (Rot. SP) copula has lower BIC than the Gumbel (Rot. IR) copula. The Clayton (Rot. IR) copula has lower BIC than the Clayton (Rot. SP) copula. This result implies that the tail dependence of a fall in the stock price and a rise in the interest rate is stronger than that of a rise in the stock price and a fall in the interest rate.

[19] Given some copula density $\hat{c}(u_1,u_2)$, the copula density with rotating interest rate axis (Rot. IR) is represented by $c(u_1,u_2)=\hat{c}(u_1,1-u_2)$, and the copula density with rotating stock price axis (Rot. SP) is represented by $c(u_1,u_2)=\hat{c}(1-u_1,u_2)$.

Table 9. MLE for the recent Spanish pseudo sample

	Parmeter	Est. Value	Std. Err.	BIC
Gumbel (Rot. IR)	γ	1.339	0.037	–142.1
Gumbel (Rot. SP)	γ	1.354	0.037	–144.9
Clayton (Rot. IR)	α	0.581	0.059	–123.5
Clayton (Rot. SP)	α	0.537	0.058	–112.0
Frank	δ	–2.554	0.231	–115.1
Gaussian	ρ	–0.419	0.027	–142.0
t	ρ	–0.403	0.034	–155.4
	ν	5.267	1.370	
Mixed-Gaussian	ρ_1	–0.531	0.035	–146.5
	ρ_2	0.703	0.132	
	θ	0.885	0.038	

Table 10 summarizes VaR and ES for the sample portfolio with the estimated parametric copulas in Table 9 and the nonparametric copula in Figure 5. The joint distribution for the risk factors is constructed by combining each copula with the recent Japan marginal distributions in Table 1. The simulation procedure for each parametric copula is the same as in Section 3.1.3.

Table 10. VaR and ES using copula estimated from the recent Spanish pseudo sample

Copula	VaR(99%)	Std. dev.	diversification effect	ES(97.5%)	Std. dev.	diversification effect
Nonparametric	4.01	––	▲21%	3.92	––	▲30%
Gumbel (Rot. IR)	3.91	0.04	▲23%	4.22	0.04	▲25%
Gumbel (Rot. SP)	4.44	0.05	▲13%	4.89	0.06	▲13%
Clayton (Rot. IR)	4.47	0.05	▲12%	4.91	0.06	▲12%
Clayton (Rot. SP)	3.68	0.04	▲27%	3.99	0.05	▲29%
Frank	3.90	0.04	▲23%	4.20	0.05	▲25%
Gaussian	4.14	0.04	▲18%	4.48	0.04	▲20%
t	4.19	0.04	▲18%	4.59	0.05	▲18%
Mixed-Gaussian	4.32	0.04	▲15%	4.67	0.05	▲17%

The diversification effect (12–30%) is much smaller than that in the recent Japanese pseudo sample in Table 7. The largest diversification effect, 30%, is observed for ES by the nonparametric copula. That result comes from the problem that the nonparametric copula does not give any probability outside the empirical range of risk factors. The same problems applies to the HS method. Let us consider a bivariate pseudo sample with size *N*. The nonparametric copula does not capture events with probability less than $1/N$; this causes underestimation of ES, which is obtained by the average loss over

VaR, especially for N that is not large.

Unlike Table 7, the diversification effect gets smaller as tail dependence gets stronger. The diversification effect of the Gumbel (Rot. SP) copula or the Clayton (Rot. IR) copula is smaller than that of the Frank copula. The diversification effect of the Gaussian copula is smaller than that of the t copula.

Unlike Table 7, the diversification effects in Gaussian, mixed-Gaussian, and t copulas are similar and small, 15–20%.

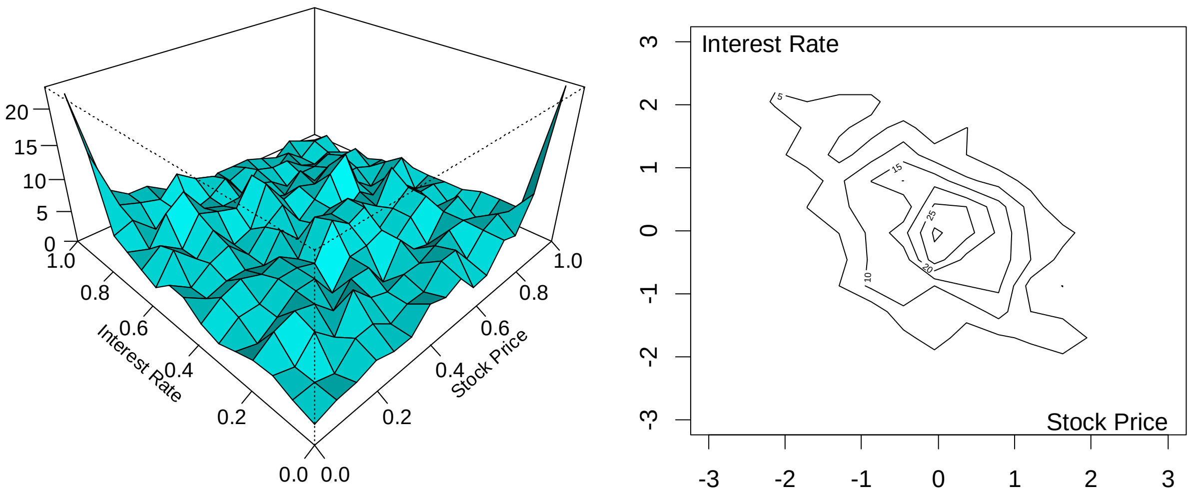
3.2.2. Italy

Adopting the FTSE MIB index as the stock price, Table 11 shows the estimated marginal distributions in separating the copulas.

Table 11. Estimated parameters and the 99 percentile point for recent Italian data

	location (μ)	scale (σ)	shape (λ)	d.f. (ν)	99 percentile
Stock price	0.005011	0.014978	–0.428	4.747	–0.05368
Interest rate	0.003873	0.052903	–0.009	1.717	0.46437

Figure 6 (a) depicts the histogram for the pseudo sample. Figure 6 (b) depicts the joint density contour with standard Gaussian margins.



(a) Histogram **(b) Contour of the joint density**

Figure 6. Joint histogram and contour plot for the recent Italian pseudo sample

Table 12 shows the result for the maximum likelihood estimation of copulas; the t copula is selected by BIC.

Table 12. MLE for the recent Italian pseudo sample

	Parmeter	Est. Value	Std. Err.	BIC
Gumbel (Rot. IR)	γ	1.400	0.039	–176.7
Gumbel (Rot. SP)	γ	1.427	0.039	–194.7
Clayton (Rot. IR)	α	0.706	0.062	–168.6
Clayton (Rot. SP)	α	0.619	0.061	–138.4
Frank	δ	–2.928	0.234	–149.9
Gaussian	ρ	–0.471	0.025	–185.0
t	ρ	–0.453	0.032	–198.7
	ν	5.019	1.303	
Mixed-Gaussian	ρ_1	–0.588	0.042	–186.5
	ρ_2	0.421	0.241	
	θ	0.855	0.066	

Table 13 summarizes VaR and ES for the sample portfolio with the estimated parametric copulas in Table 12 and the nonparametric copula in Figure 6.

Table 13. VaR and ES using copula estimated from the recent Italian pseudo sample

Copula	VaR(99%)	Std. dev.	diversification effect	ES(97.5%)	Std. dev.	diversification effect
Nonparametric	3.82	––	▲25%	3.91	––	▲30%
Gumbel (Rot. IR)	3.97	0.04	▲22%	4.28	0.05	▲23%
Gumbel (Rot. SP)	4.53	0.06	▲11%	4.99	0.07	▲11%
Clayton (Rot. IR)	4.57	0.06	▲10%	5.03	0.07	▲10%
Clayton (Rot. SP)	3.71	0.04	▲27%	4.02	0.05	▲28%
Frank	3.96	0.04	▲22%	4.25	0.04	▲24%
Gaussian	4.22	0.04	▲17%	4.57	0.04	▲18%
t	4.27	0.05	▲16%	4.68	0.05	▲16%
Mixed-Gaussian	4.42	0.04	▲13%	4.79	0.06	▲14%

The results in Table 13 are almost the same as those for the Spanish pseudo sample shown in Table 10. However, VaR by the nonparametric copula in Table 13 is slightly different from that in Table 10, which indicates that even the same amount of data with a similar situation have different *k*-th maximum loss value for a given *k*. VaR with a nonparametric copula is much more dependent on the given data than the VaR with a parametric copula.

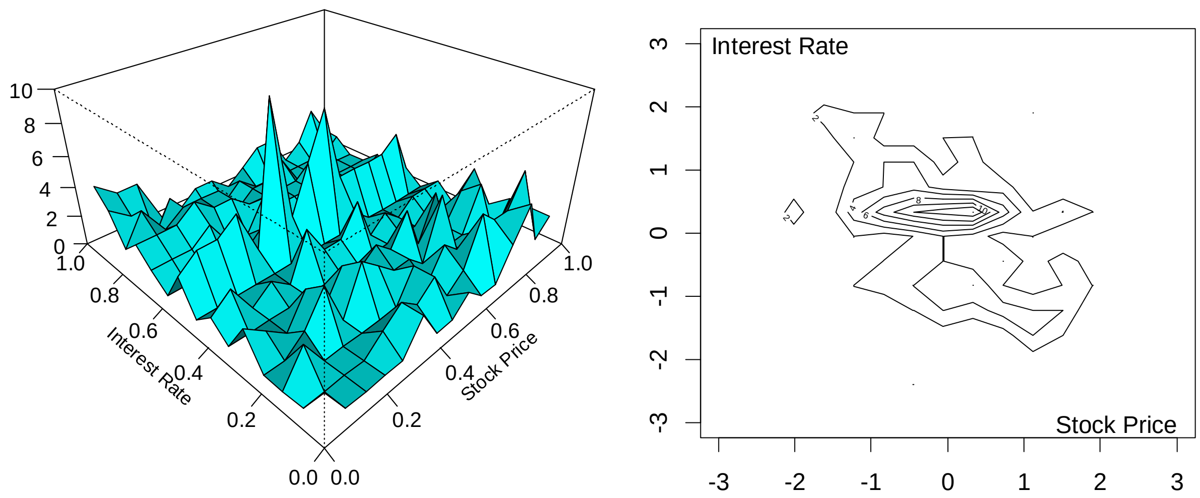
3.3. Post-Bubble data in Japan

Table 14 shows the marginal distributions in separating the copulas estimated by one-year daily data in FY1990 (from April 1, 1990 to March 31, 1991) in Japan.

Table 14. Estimated parameters and the 99 percentile point for FY1990 Japanese data

	location (μ)	scale (σ)	shape (λ)	d.f. (ν)	99 percentile
Stock price	–0.001318	0.014526	0.045	3.963	–0.05481
Interest rate	–0.000070	0.000255	0.118	1.583	0.00273

Figure 7 (a) depicts the histogram for the pseudo sample. Figure 7 (b) depicts the joint density contour with standard Gaussian margins. We see a negative linear correlation between the stock price and the interest rate in FY1990, the post-Bubble period in Japan.



(a) Histogram **(b) Contour of the joint density**

Figure 7. Joint histogram and contour plot for the FY1990 Japanese pseudo sample

Table 15 is the result of the maximum likelihood estimation of copulas. Similar to the Spanish pseudo sample (Table 9) and the Italian pseudo sample (Table 12), the *t* copula is selected by BIC.

Table 15. MLE for the FY1990 Japanese pseudo sample

	Parmeter	Est. Value	Std. Err.	BIC
Gumbel (Rot. IR)	γ	1.285	0.063	–24.1
Gumbel (Rot. SP)	γ	1.285	0.063	–25.5
Clayton (Rot. IR)	α	0.422	0.103	–16.3
Clayton (Rot. SP)	α	0.448	0.103	–18.7
Frank	δ	–2.489	0.419	–30.1
Gaussian	ρ	–0.315	0.055	–19.9
t	ρ	–0.378	0.061	–31.4
	ν	3.802	1.084	
Mixed-Gaussian	ρ_1	–0.707	0.067	–29.6
	ρ_2	0.237	0.177	
	θ	0.635	0.112	

Table 16 summarizes VaR and ES for the sample portfolio with the estimated parametric copulas in Table 15 and the nonparametric copula in Figure 7.

Similar to the Spanish pseudo sample (Table 10) and the Italian pseudo sample (Table 13), the diversification effects for parametric copulas in Table 16 (10–29%) are much smaller than those in the recent Japanese pseudo sample (Table 7). The diversification effect for the nonparametric copula is larger than that for each parametric copula, which reflects the small size of the sample.

Table 16. VaR and ES using copula estimated from the FY1990 Japanese pseudo sample

Copula	VaR(99%)	Std. dev.	diversification effect	ES(97.5%)	Std. dev.	diversification effect
Nonparametric	3.37	––	▲34%	3.57	––	▲36%
Gumbel (Rot. IR)	3.85	0.03	▲24%	4.16	0.04	▲26%
Gumbel (Rot. SP)	4.33	0.04	▲15%	4.77	0.06	▲15%
Clayton (Rot. IR)	4.29	0.05	▲16%	4.72	0.06	▲16%
Clayton (Rot. SP)	3.66	0.04	▲28%	3.96	0.05	▲29%
Frank	3.90	0.04	▲23%	4.19	0.05	▲25%
Gaussian	3.98	0.03	▲22%	4.31	0.04	▲23%
t	4.17	0.05	▲18%	4.58	0.06	▲18%
Mixed-Gaussian	4.58	0.05	▲10%	4.98	0.06	▲11%

The diversification effect for the selected *t* copula for the Spanish pseudo sample (Table 10), the Italian pseudo sample (Table 13), and the post-Bubble Japan pseudo sample (Table 16) varies from 16% to 18%, much smaller than that for *t* copula for the recent Japanese data (Table 7), 49%.

3.4. Implications

The results of the data analysis in this section are summarized as follows.

First, the VCV method tends to underestimate portfolio risk because it adopts Gaussian distribution that cannot capture the fat-tail properties of profit–loss distributions.

Second, a copula extracts the dependence structure from stressed markets. Combining the copula with the marginal distributions estimated from the recent data can simulate a stressed dependence situation that has not been observed in the recent data.

Third, a nonparametric copula has problems in estimating ES and VaR: underestimation of ES and strong data dependency of VaR.

Fourth, the *t* copula or the mixed-Gaussian copula tends to be selected by BIC, which implies that the dependency between the stock price and the interest rate cannot be captured by a linear correlation.

Fifth, strong tail dependence is not necessarily conservative. It sometimes increases the risk reduction of the portfolio.

4. Conclusions and Open Problems

We have discussed how to incorporate a stressed situation into risk measurement by using a copula. This section addresses the applicability of a copula in financial practice and states the open problems.

First, a bivariate or small-number-of-risk-factors approach has been utilized in financial practice. As shown in this paper, some financial institutions aggregate risks of stocks and bonds represented by the key interest rate and the stock index even if they measure the sub-portfolio risks by using more risk factors. Some major financial institutions aggregate the firm-wide risks of market, credit, and operational risk categories by representing each risk category with some profit–loss time series (see Brockmann and Kalkbrener [2010], for example). They sometimes calibrate Gaussian or *t* copula parameters (including ν for *t* copula) by using one time series each for market and credit profit–loss, and determine the correlation with other risk categories a priori without statistical estimation.

Second, the bivariate approach in this paper can be extended to more than two risk factors. For Gaussian, *t*, and mixed-Gaussian copulas, multivariate extension can be done by replacing a correlation parameter ρ with a correlation matrix Ω. In practice, various simplifications are done in estimation of parameters. For example, a correlation matrix Ω for Gaussian or *t* copula is sometimes estimated from the sample rank correlation.[20] Parameter ν for *t* copula is sometimes estimated from the joint tail frequency.[21] One parameter, Archimedean copula including a Gumbel, Clayton, or Frank copula can be easily extended to more than two variables if homogenous pair-wise dependency can be assumed. To incorporate various pair-wise dependency, HAC (hierarchal Archimedean copula)[22] or vine copula[23] may be needed. Skew-*t* copula is another way to incorporate asymmetric dependency.[24] Simplifications of estimation for HAC, vine, and skew-*t* copula are open problems in practice.

Third, selecting a suitable copula for risk factors may strongly depend on the choice of approach: either "unconditional" or "conditional." We adopt an unconditional approach because most financial institutions calculate daily VaR to estimate the necessary capital to cover the potential loss for one day under an unconditional approach. The loss will not be suffered the next day, but on some day during the capital planning year with a small probability. On the other hand, most academic research focuses on a conditional approach to estimate daily VaR by using all available information at evaluation.[25] A conditional approach applies a copula to unpredictable parts in returns by some time-series model such as GARCH. Although an unconditional approach may need a flexible and complicated copula, a conditional approach may only need a simple copula.

---

20 Pairwise rank correlation including Spearman rho and Kendall tau has one-to-one correspondence to pairwise correlation ρ_{ij} in Gaussian or *t* copula. See McNeil, Frey, and Embrechts [2005]. For mixed-Gaussian copula with more than two states or variables, EM algorithm can be applied for maximum likelihood estimation. However, more efficient or robust estimation is an open problem.

21 For any pair of variables (*i*, *j*) with *t* copula, the conditional probability *j*-th variable is less than *u*, given that the *i*-th variable is less than *u* and converges to $2t_{\nu+1}(-\sqrt{(1-\rho_{ij})(\nu+1)/(1+\rho_{ij})})$. Since ρ_{ij} is already fixed as footnote 20, equating that probability to the joint tail frequency divided by some small threshold *u* estimates the parameter ν.

22 For HAC, see Savu and Trede [2010], for example.

23 For vine copula, see the handbook of Kurowicka and Joe [2010].

24 Demarta and McNeil [2005] and Smith, Gan, and Kohn [2012] propose various types of skew-*t* copula.

25 See Christoffersen *et al.* [2012] and Smith, Gan, and Kohn [2012], for example.

Finally, adjustment of the holding period is a significant open problem that has not been discussed in this paper. If the holding period is longer than the data frequency, then we have to calculate the cumulative profit–loss distribution. The calculation has two issues even if the observed sample has no serial correlation under an unconditional approach. The first theoretical issue is the conversion speed of the tail in the cumulative profit–loss distribution. Fushiya and Kusuoka [2010] indicate that the conversion speed to the Gaussian tail is slow if the unit profit–loss distribution is fat even with a finite variance. The second practical issue lies in choosing a method to determine the reduction rate of the position by considering market liquidity (see Brockmann and Kalkbrener [2010], for example).

References

Aas, Kjersti and Ingrid Hobaek Haff, "The Generalized Hyperbolic Skew Student's *t*-Distribution," *Journal of Financial Econometrics*, **4**(2), 2006, pp. 275–309.

Azzalini, Adelchi and Antonella Capitanio, "Distributions generated by perturbation of symmetry with emphasis on a multivariate skew *t*-distribution," *Journal of the Royal Statistical Society Series B*, **65**(2), 2003, pp. 367–389.

Basel Committee on Banking Supervision, Joint Forum "Developments in Modelling Risk Aggregation," 21 October, 2010.

Brockmann, Michael and Michael Kalkbrener, "On the aggregation of risk," *Journal of Risk*, **12**(3), 2010, pp. 45–68.

Christoffersen, Peter, Vihang Errunza, Kris Jacobs, and Hugues Langlois, "Is the Potential for International Diversification Disappearing? A Dynamic Copula Approach," *Review of Financial Studies*, **25**(12), 2012, pp. 3711–3751.

Daníelsson, Jón, Bjørn N. Jorgensen, Gennady Samorodnitsky, Mandira Sarma, and Casper G. de Vries, "Fat Tails, VaR and Subadditivity," *Journal of Econometrics*, **172**(2), 2013, pp. 283–291.

Demarta, Stefano and Alexander J. McNeil, "The *t* Copula and Related Copulas," *International Statistical Review*, **73**(1), 2005, pp. 111–129.

Fernández, Carmen and Mark F. J. Steel, "On Bayesian Modeling of Fat Tails and Skewness," *Journal of the American Statistical Association*, **93**(441), 1998, pp. 359–371.

Fukuda, Yoshiyuki, Kazutoshi Kan, and Yoshihiko Sugihara, "Banks' Stockholdings and the Correlation between Bonds and Stocks: A Portfolio Theoretic Approach," Bank of Japan Working Paper Series No.13-E-6, 2013.

Fushiya, Hirotaka and Shigeo Kusuoka, "Uniform Estimate for Distributions of the Sum of i.i.d. Random Variables with Fat Tail," *Journal of Mathematical Sciences*, University of Tokyo, **17**(1), 2010, pp. 79–121.

Hansen, Bruce, "Autoregressive conditional density estimation," *International Economic Review*, **35**(3), 1994, pp. 705–729.

Isogai, Takashi, "Benchmarking unconditional VaR and ES calculation methods: a numerical simulation with fitted truncated stable distribution," Bank of Japan Working Paper Series, 2013, forthcoming.

Joe, Harry, *Multivariate Models and Dependence Concepts*, Chapman & Hall, 1997.

Kojadinovic, Ivan, Jun Yan, and Mark Holmes, "Fast large-sample goodness-of-fit tests for copulas," *Statistica Sinica*, **21**(2), 2011, pp. 841–871.

Kubota, Kocho, "Issues on Market VaR Calculation and Improvement by Parametric Method," IMES Discussion Paper Series No. 2009-J-27, Bank of Japan, 2009 (in Japanese).

Kurowicka, Dorota and Harry Joe, *Dependence Modeling: Vine Copula Handbook*, World Scientific Publishing, 2010.

McNeil, Alexander J., Rüdiger Frey, and Paul Embrechts, *Quantitative Risk Management: Concepts, Techniques, and Tools*, Princeton University Press, 2005.

Miura, Ryozo and Shingo Oue, "Statistical Methodologies for the Market Risk Measurement," *Asia-Pacific Financial Markets*, **7**(4), 2000, pp. 305−319.

Nelsen, Roger B., *An Introduction to Copulas*, 2nd edition, Springer, 2006.

Savu, Cornelia and Mark Trede, "Hierarchies of Archimedean copulas," *Quantitative Finance*, **10**(3), 2010, pp. 295–304.

Smith, Michael S., Quan Gan, and Robert J. Kohn, "Modelling dependence using skew *t* copulas: Bayesian inference and applications," *Journal of Applied Econometrics*, **27**(3), 2012, pp. 500–522.