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Extracting Market Expectations on the Duration of the Zero Interest Rate Policy from Japan's Bond Prices

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Abstract

This paper aims to extract the expectations of market participants on the duration of the Zero Interest Rate Policy (ZIRP) by the Bank of Japan by modeling the term structure of interest rates. Under the ZIRP, particularly the short-term and medium-term interest rates are so low that we face difficulty applying traditional yield curve models such as the Vasicek model to them. This circumstance motivated us to model the expectations of market participants on the duration of the ZIRP by regarding it as one of the risk factors of a yield curve. To be specific, we constructed a yield curve model with the following two interest-rate generating process by maturity zone: (i) the short-term interest rate zone follows the traditional Vasicek model augmented by incorporating the probability of policy duration as one of the risk factors, and (ii) the long-term interest rate zone is determined by risk prices perceived in markets. By taking these steps, we significantly improved the fitting of the yield curve model under the ZIRP.

Key Words: ZIRP; Policy Duration; Term Structure Model; Vasicek Model

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1. Introduction

This paper aims to analyze the properties of "policy duration effects," using Japanese yield curve data in the two periods during which the uncollateralized overnight call rate fluctuated almost at zero percent[1]. We define "policy duration effects" as those that lower the medium-term interest rates through expectations of market participants on how long the Bank of Japan (BOJ) will carry on the zero interest rate policy.

On February 12, 1999, the BOJ made the following decision: (i) the BOJ would encourage the uncollateralized overnight call rate to move as low as possible by providing ample liquidity; (ii) to avoid excessive volatility in the short-term financial markets, the BOJ would, by paying due attention to maintaining the sound market functioning, initially aim to guide the call rate to move around 0.15 percent, and subsequently induce a further decline in line with the market developments. This is the beginning of the so-called "Zero Interest Rate Policy (ZIRP)". Following the decision, the BOJ gradually lowered the target rate to almost zero percent. Moreover, on April 13, 1999, the BOJ governor Hayami stated that the BOJ would continue the ZIRP until deflationary pressures would be dispelled.

Although the policy was once abandoned in August 2000, on March 19, 2001, the BOJ decided to reactivate the policy in the following enhanced manner: (i) the BOJ switched the operating target for money market operations from the uncollateralized overnight call rate to the outstanding balance of the current accounts held by financial institutions at the BOJ; (ii) the BOJ carries on the new policy framework until the consumer price index registers a zero percent or more on a yearly basis. Shortly after this, the uncollateralized overnight call rate came down to almost zero percent again. This policy framework is called "quantitative monetary easing policy."

Under these circumstances, we face difficulty fitting traditional yield curve models, including the Vasicek model, to Japan's bond prices. It is due to the fact that under the ZIRP Japan's yield curve has a peculiar shape particularly in the short-term and medium-term zones. Put differently, interest rates in these zones were extremely low reflecting the expectations of market participants on the duration of the ZIRP. In general, these effects are called "policy duration effects." For example, Shiratsuka and Fujiki (2001) showed that the BOJ's commitment to the future path of monetary policy

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[1] We generically define the following two periods as the periods under the ZIRP: (i) from February 1999 to August 2000, and (ii) from March 2001 to the present. The former period corresponds to that under the ZIRP, and the latter period corresponds to that under the quantitative monetary easing policy.

contributed to flattening and stabilizing yield curves to a large extent by analyzing the term structure of instantaneous forward rates.

This paper aims to overcome the difficulty by augmenting the traditional yield curve model with a random variable that captures the expectations of market participants on the policy duration. Here, we define the period from the present to when the ZIRP is abandoned as the period of policy duration, treating it as a random variable following a certain probability distribution. Also, we assume that an instantaneous spot rate is zero under the ZIRP, while it follows the Vasicek model after the BOJ abandons the ZIRP.

Based on these assumptions, our model attempts to extract the probability of policy duration from Japan's bond price data. Findings in this paper are summarized as follows: (i) our model significantly improved the fitting of the yield curve model, particularly in short and medium-term rate zones, compared with the traditional Vasicek model; (ii) the estimated results go well with the anecdotal episode, particularly when the BOJ once ended the ZIRP in August 2000.

The rest of the paper is organized as follows. In section 2, we first explain an outline and operational problems in applying the Vasicek model to Japan's yield curve data under the ZIRP. Then, we augment the traditional Vasicek model with the probability of policy duration and describe the estimation method of the parameters in the model. In section 3, we estimate the parameters and derive the expected policy duration. Finally, in section 4, we conclude this paper.

2. Yield Curve Models under the ZIRP

In general, yield curve models can be divided into: (i) spot rate models and (ii) forward rate models. Spot rate models such as the Vasicek model[2], the CIR model[3], and the Hull and White model[4], formulate zero coupon bond prices, regarding risk-free zero coupon bonds as derivatives whose state variable is an instantaneous spot rate[5]. In this

[2] See Vasicek (1977).
[3] See Cox, Ingersoll, and Ross (1985).
[4] See Hull and White (1990).
[5] An instantaneous spot rate is an imaginary rate in modeling the term structure of interest rates. By imaginary we mean that it is not observable in the markets. Instinctively, it denotes a return of a risk-free asset with marginal maturities. In practice, a money market rate or an intersection at a yield curve to the vertical line corresponding to the zero remaining maturity is often used as proxy for an instantaneous spot rate (see Yoshiba and Ieda (2001)). This issue is discussed in detail in the following section.

type of model, the shape of a yield curve is dependent only on fluctuations in instantaneous spot rates. On the other hand, forward rate models, such as the HJM model[6], construct a yield curve by specifying forward rates with the corresponding maturities.

Among these various yield curve models, we focus on the Vasicek model, which assumes that an instantaneous spot rate r_t at time t follows the mean-reverting process:

$$dr_t = \kappa(m - r_t)dt + \sigma dW_t, \tag{1}$$

where W_t is the Wiener process, and κ, m, σ are positive parameters. Also, $v(t,T)$, risk-free zero coupon bond prices at time t with maturity T, can be written as

$$v(t,T) = H_1(T-t)e^{-H_2(T-t)r(t)}, \tag{2}$$

$$H_1(t) = \exp\left\{\frac{(H_2(t)-t)(\kappa^2\mu - \sigma^2/2)}{\kappa^2} - \frac{\sigma^2 H_2^2(t)}{4\kappa}\right\}.$$

$$H_2(t) = \frac{1-e^{-\kappa t}}{\kappa}.$$

The Vasicek model has some desirable properties such as a mean-reverting property. In many cases, it has simple analytical solutions to pricing problems of derivatives. On the other hand, it has some undesirable properties such as allowing negative interest rates, as well as assuming a time-invariant volatility. Moreover, the yield curves constructed employing the Vasicek model are upper convex in shape no matter what values the parameters take. These properties indicate that the yield curves thus constructed are inappropriate in the period of the ZIRP, during which they are lower convex in shape in short-term and medium-term zones.

In what follows, we attempt to modify the traditional Vasicek model of yield curves by incorporating a random variable for policy duration to improve the fitting of the model to actual interest rates across yield curves under the ZIRP.

2.1 Dynamics of Instantaneous Spot Rates

We define the period until the BOJ ends the ZIRP as a random variable τ, while P and P^* as an original probability measure and an equivalent martingale

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[6] See Heath, Jarrow, and Morton (1992).

measure with respect to P, so-called a risk neutral probability measure, respectively. Also, we define the probability space with a filtration as $(\Omega, F, \{F_t\}, P)$[7]. We write indicator function $1_{\{\cdot\}}$ as follows:

$$1_{\{A\}}(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases} , \tag{3}$$

where τ is specified as a stopping time whose stocastic process $\{1_{\{\tau \le t\}}\}$ is $\{F_t\}$-adaptive. Since short-term rates move almost at zero percent under the ZIRP, we assume that a spot rate is zero under the ZIRP and follows the Vasicek model once the policy is abandoned.

Therefore, under the equivalent martingale measure P^*, an instantaneous spot rate at time t given τ, r_t^τ satisfies

$$dr_t^\tau = 1_{\{\tau \le t\}} \left\{ \kappa \left(\mu - r_t^\tau \right) dt + \sigma \, dW_t^* \right\}, \tag{4}$$

where τ and W^* are assumed to be independent under P^*. Also, the relationship between W under the original probability measure P in equation (1) and W^* under P^* in equation (4)[8] enables us to rewrite drift μ in equation (4) as

$$\mu = m - \frac{\sigma}{\kappa} \lambda , \tag{5}$$

where λ is the market price of risk[9].

2.2 Zero Coupon Bond Prices and Yield Curves

Next, we derive a yield curve model under the ZIRP. $v(t,T)$ is expressed as

$$v(t,T) = E^{P^*} \left[\exp \left\{ - \int_t^T r_s^\tau \, ds \right\} \middle| F_t \right], \tag{6}$$

[7] See Morimura and Kijima (1991).
[8] See Kijima (1999).
[9] See Appendix 1.

where $E[\cdot]$ denotes an expectation operator under P^*. Under the condition $\tau > t$, we can rewrite equation (6) as follows, using equation (4)[10].

$$v(t,T) = \int \frac{\psi(s)}{1-\psi(t)} H_1(T-s)ds + \frac{1-\Psi(T)}{1-\Psi(t)}, \tag{7}$$

$$H_1(t) = \exp\left\{ \frac{(H_2(t)-t)(\kappa^2\mu - \sigma^2/2)}{\kappa^2} - \frac{\sigma^2 H_2^2(t)}{4\kappa} \right\},$$

$$H_2(t) = \frac{1-e^{-\kappa t}}{\kappa},$$

where ψ and Ψ are defined as a probability density function and a probability distribution function of τ, respectively. Then, yield $Y(t,T)$ can be written as

$$Y(t,T) = -\frac{\ln v(t,T)}{T-t}. \tag{8}$$

The estimation method of parameters in (8) is discussed in the next section.

3. Empirical Analysis

This section evaluates the appropriateness of our model from the following two perspectives. First, we evaluate the fitting of the model by comparing between the interest rates observed in the markets and those predicted by the model. To this end, we need to estimate three parameters, κ, m, and σ, of the Vasicek model from Japanese government bond (JGB) prices. Second, we evaluate the estimated probability distribution about the market participants' expectations on the duration of the ZIRP in terms of relevance to the reality.

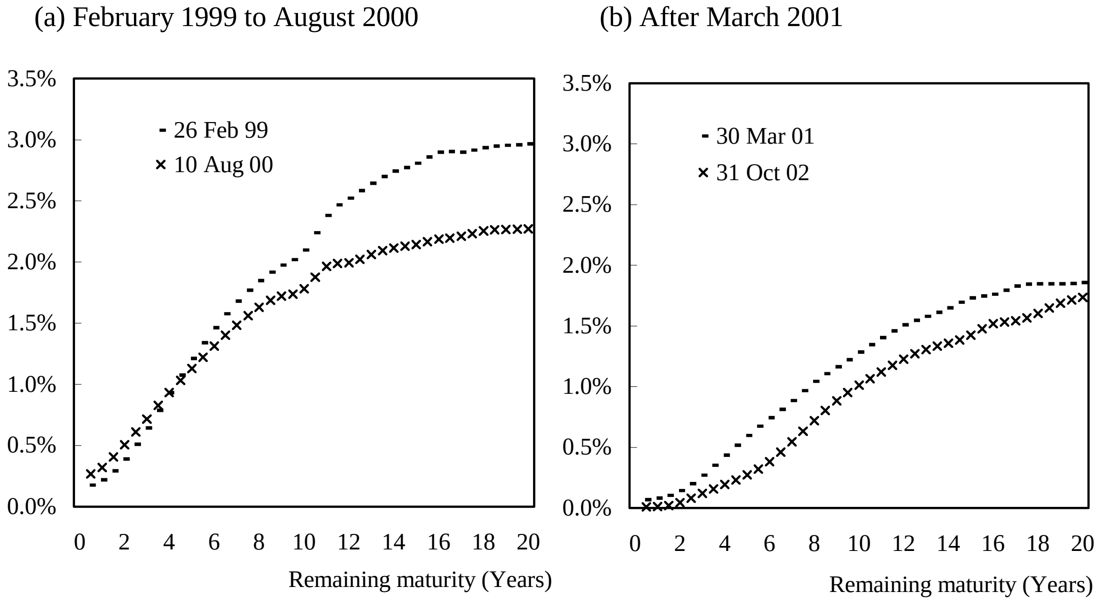
3.1 Estimation of Instantaneous Spot Rates and Parameters of the Vasicek Model

In our model, a yield curve is defined in terms of the prices of discount bonds. In JGB markets, however, coupon bonds account for a large portion of amounts outstanding, while the issuance of discount bonds has been relatively small. Thus, market liquidity of discount bonds is quite low compared with that of coupon bonds.

This feature makes us consider using the prices of coupon bonds in place of the prices of discount bonds in estimating yield curves. To be specific, we need to estimate discount bond prices using coupon bond prices by assuming one-to-one correspondence between the two. We employ the McCulloch model[11] using the price data of coupon bonds with 5-, 10- and 20-year remaining maturity. We exclude newly issued bonds due to their extremely high liquidity. The data we use is "Over-the-Counter Standard Bond Quotations" released by the Japan Securities Dealers Association.

Figure 1 shows examples of the yield curves estimated by the above method. In what follows, we express yield curves per 0.5-year remaining maturity and call the estimated yield curves market interest rates for simplicity.

Figure 1: Examples of Estimated Yield Curves



Estimation of the Vasicek model needs a time series of instantaneous spot rates. Since instantaneous spot rates can be defined only in theoretical terms and thus are not observable in real markets, we need to find a proxy for them. Sometimes, money market rates such as call rates are used as the proxy. However, we chose to estimate

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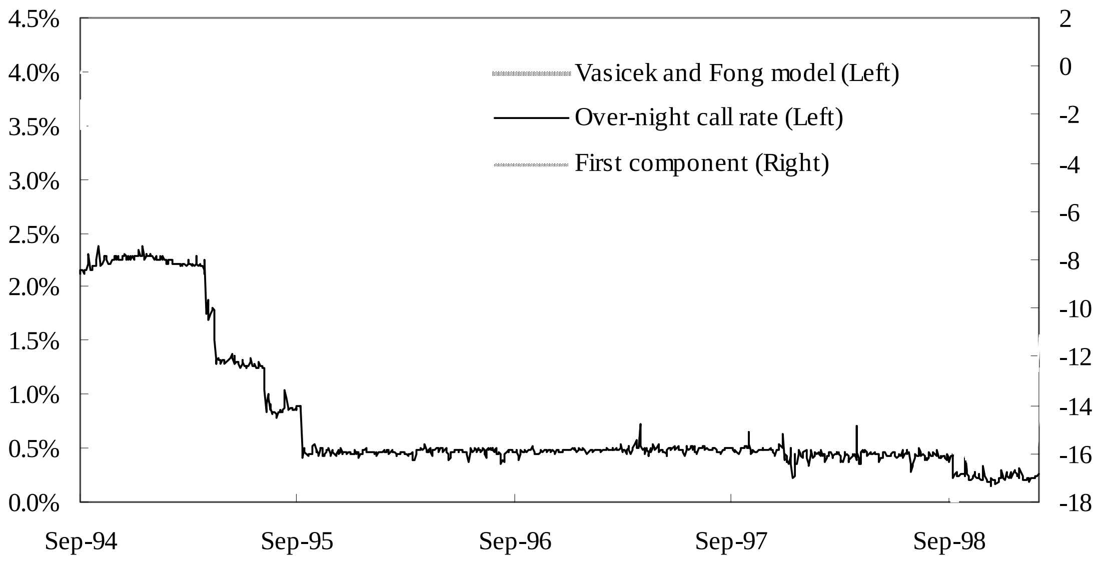
[10] See Appendix 2.
[11] See Appendix 3 or McCulloch (1971) for details.

instantaneous spot rates using bond prices obtained from the Vasicek and Fong model[12] for the following reason.

Principal component analysis for the yield curves on a daily basis shows that the first component is dominant[13]. Thus, the first principal component can approximately follow instantaneous spot rates in a one-factor spot rate model, such as the Vasicek model[14]. Figure 2 shows instantaneous spot rates estimated by the Vasicek and Fong model, the accumulated value of the first principal component, and uncollateralized over-night call rates. This figure indicates that instantaneous spot rates estimated by the Vasicek and Fong model are more strongly correlated with the accumulated value of the first principal component than call rates. It also shows the low correlation between over-night call rates and the first principal component. Therefore, we consider the instantaneous spot rates estimated by the Vasicek and Fong model more suitable than call rates as instantaneous spot rates in our model.

Figure 2: Comparison between the Instantaneous Spot Rates Estimated by the Vasicek and Fong Model and the Accumulated Value of the First Component of Principal Component Analysis



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[12] Appendix 5 explains the procedure of deriving instantaneous spot rates by the Vasicek and Fong model.

[13] Our principal component analysis, applying to interest rates before the ZIRP was adopted, shows that the first principal component explains 89% of the total fluctuation in the yield curve. In general, the first principal component is considered to reflect the parallel shift of yield curves.

[14] For the computing method of the first principal component, see Appendix 4.

Let us estimate the parameters of the Vasicek model. As a preparatory step, we rewrite equation (1), which is written in a continuous-time setting, to that in a discrete-time setting since the highest frequency with which we can estimate instantaneous spot rates by the Vasicek and Fong model is a daily basis.

Changing the infinite short time interval dt to the finite time interval Δ in equation (1) yields $dr_t \to r_{t+\Delta} - r_t$ and $\sigma dW_t \to \sigma\sqrt{\Delta}u_t$, where u_t denotes the random variable following the standard Gaussian distribution. Thus, equation (1) can be written as

$$r_{t+\Delta} - r_t = \kappa(m - r_t)\Delta + \sigma\sqrt{\Delta u_t} \ . \qquad (9)$$

We estimate the parameters m, κ, σ in equation (9) employing the maximum likelihood method with Δ set as 1/250.

Table 1 shows the estimation result of the parameters. It should be noted here that κ, which denotes the speed of mean-reversion of the instantaneous spot rates, is not stable across the sample periods. In what follows, we use the parameters estimated using the longest sample period i.e., from January 1992 to January 1999, as the baseline case since the Vasicek model assumes the long-term process of instantaneous spot rates.

Table 1: Estimation Results of the Parameters of the Vasicek Model

			κ	m	σ
06 Jan 92	–	29 Jan 99	0.7131 (1.974)**	0.006476 (0.8477)	0.01017 (89.40)***
04 Jan 95	–	29 Jan 99	2.329 (3.737)***	0.004061 (1.593)	0.009323 (77.03)***
06 Jan 97	–	29 Jan 99	6.123 (3.000)***	0.002672 (3.002)***	0.007224 (48.35)***

Figures in the parentheses denote t-value. * denotes 90%, ** 95%, and *** 99% confidence level, respectively.

3.2 Estimation of the Expectations of Market Participants on the Duration of the ZIRP

In this section, we estimate the market price of risk λ and the probability distribution function of the expectations of market participants on the duration of the ZIRP in equation (7). In estimating the parameter of probability distribution, we need to specify the probability distribution function that τ, the duration of the ZIRP, follows.

We can list some of the desirable properties that the probability distribution should have: unimodal, non-negativity and parsimonious functional form. Some probability distributions such as the gamma distribution, the exponential distribution and the Weibull distribution satisfy these properties. It is natural to think that the estimation results depend on the specification of the probability distribution function. In this paper, we adopt the standard gamma distribution to describe the expectations of market participants since the probability distribution function is mathematically simple; besides, it gives reasonable results compared with those from other probability distribution functions[15].

The probability density function of the standard gamma distribution is written as

$$P^*(\tau \le t) = \Psi(t) = \frac{\Gamma_t(\alpha)}{\Gamma(\alpha)}\ , \qquad \alpha > 0 \qquad (10)$$

$$\Gamma_t(\alpha) = \int_0^t u^{\alpha-1} e^{-u}\, du\ ,$$

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du\ ,$$

where α is a parameter. When $\alpha = 1$, which means the average of expectations on the ZIRP duration is 1 year, this distribution is equivalent to the exponential distribution.

Since we adopt the standard gamma distribution, our model has two parameters to be estimated: the parameter in the standard gamma distribution α and the market price of risk λ. We estimate these parameters by regressing the model to market interest rates employing the ordinary least-squares (OLS) method.

Figure 3 compares between the yield curves estimated by our model and the Vasicek model[16]. Although these models have the same property as the one-factor spot rate model, our model is more appropriate to describe market interest rates than the Vasicek model judging from the fact that the residual of our model is much less than that of the Vasicek model. The time series of residual between the predicted value of our model and market interest rates (Figure 4) show that the fitting of our model is satisfactory across any remaining maturity. The sum of the residual squares is also lower when our model is applied.

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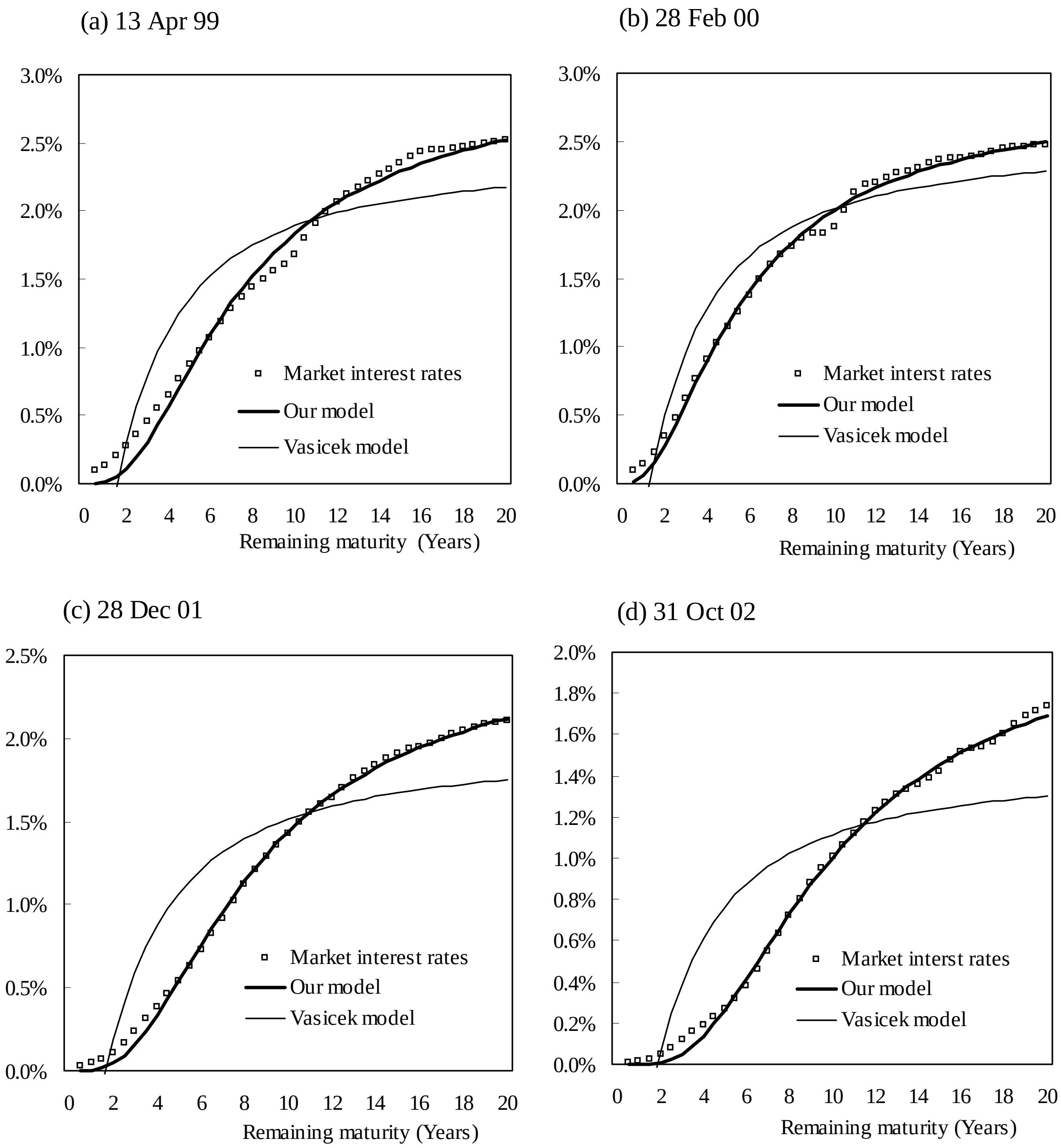
[15] See appendix 6 for the estimation results when we adopt other distributions.
[16] In applying the Vasicek model to market interest rates, we set the instantaneous spot rate and the parameters in mean-reverting process of instantaneous spot rates at values shown in the previous

Figure 3: Estimated Yield Curves:
A Comparison Between Our Model and the Vasicek Model



The estimated value of α shows the exact form of the probability distribution of expectations on the duration of the ZIRP. Figure 5 shows examples of the estimated distribution. In the period before August 2000, when the ZIRP was abandoned by the BOJ, the peaks of the probability density functions shift leftward toward the end of the ZIRP. In the period after March 2001, when the quantitative monetary easing policy was adopted, the peak shifts rightward as time passes.

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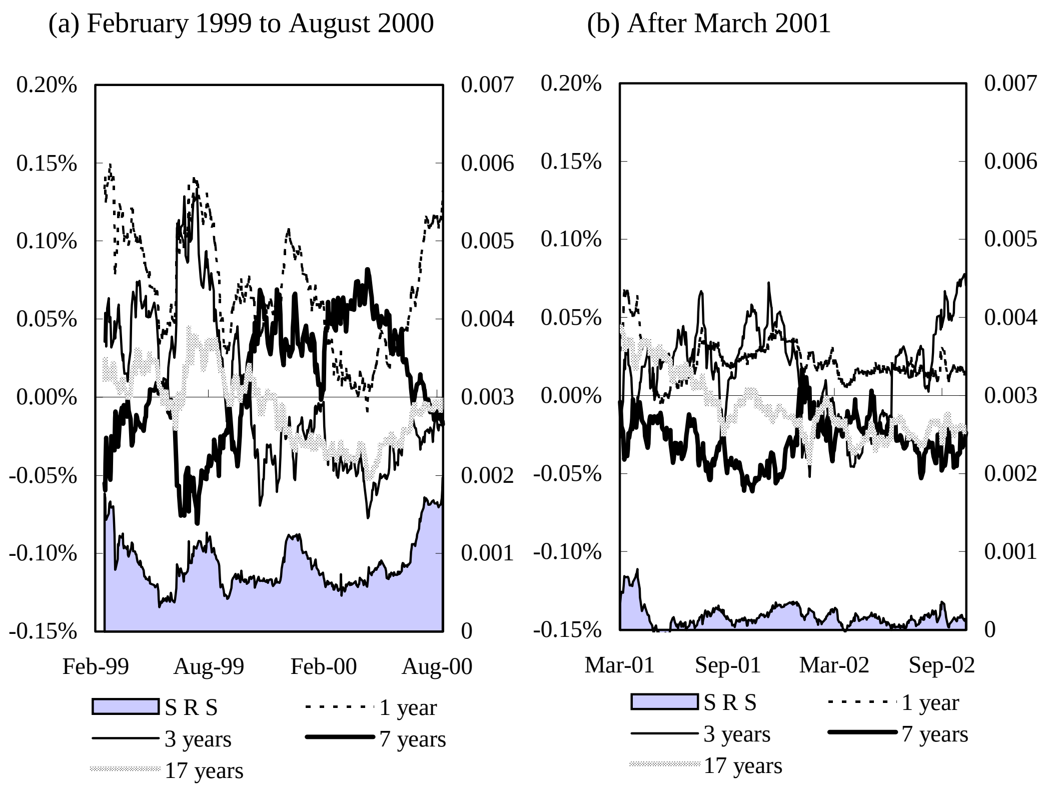
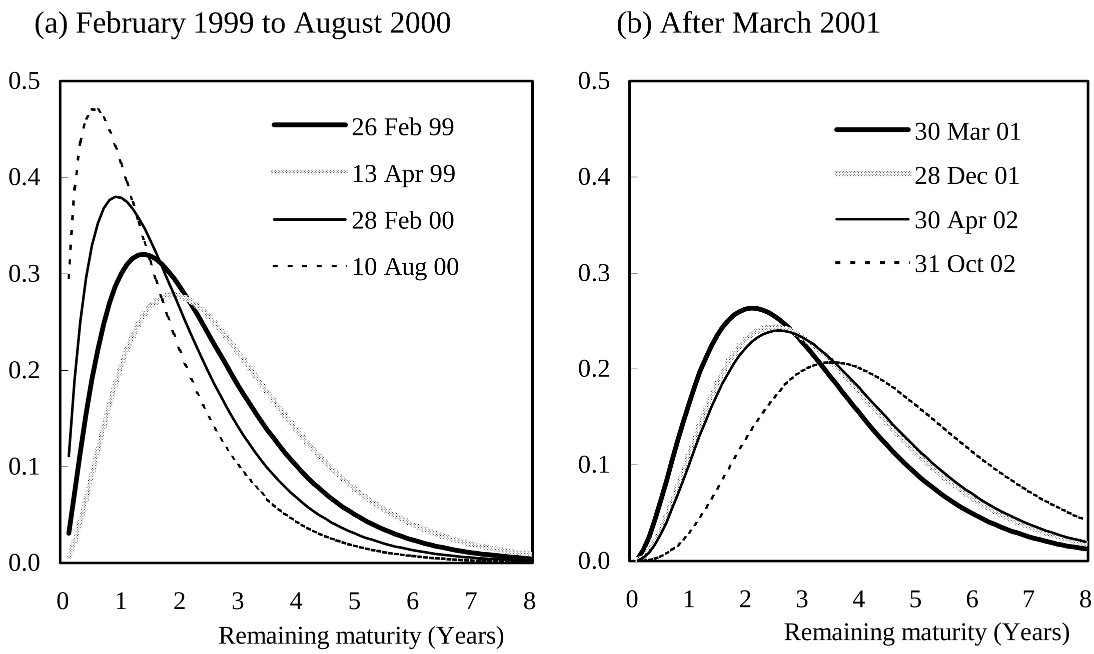
section. We estimate the market price of risk employing the OLS method.

Figure 4: Residual of Our Model



The right scale shows the sum of residual squares (S R S) and the left scale does the others.

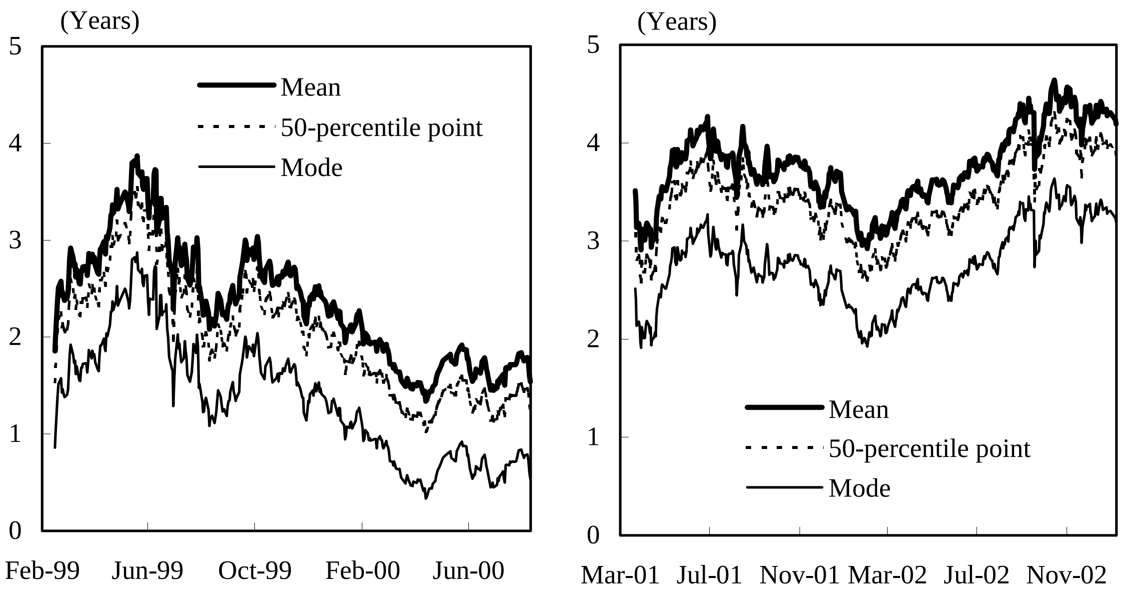
Figure 5: Examples of Estimated Probability Density Functions



To evaluate the expectations on policy duration, we need to represent the probability distribution function by some statistics. Figure 6 shows the time series of mean, mode and 50-percentile of the estimated probability density functions. Although the level of the expected policy duration depends on the statistics we choose[17], the transition point in time perceived by market participants seems to be reasonable judging from the observation that the level of each statistic lowers toward the end of the ZIRP.

Figure 6: Time Series of Mean, Mode, and 50-percentile of Estimated Probability Density Functions

(a) February 1999 to August 2000 (b) After March 2001



4. Conclusion

This paper constructed a yield curve model that is applicable to the period of the ZIRP. To be more specific, we augmented the traditional Vasicek model by incorporating a random variable that captures the expectations of market participants on policy duration. Empirical results show that our model significantly improved the fitting of the yield curve model, compared with the traditional Vasicek model. Moreover,

[17] The parameters in the Vasicek model also influence expected policy duration. For instance, when we use the parameters estimated from another sample period, from January 1997 to January 1999, estimated policy duration becomes longer by about 1 year than that shown in Figure 6. The direction of the transition remains unchanged, however.

estimated results about policy duration go well with the anecdotal episode when the BOJ once abandoned the ZIRP in August 2000.

Finally, lets us conclude the paper by listing the problems left in this paper. First, since we needed to assume the probability distribution of the ZIRP duration a priori, we used as many types of distribution as possible in order to exclude arbitrariness. But, whether this approach is appropriate should be tested more rigorously. Second, although we estimated an instantaneous spot rate using only government bond data under the assumption that a perfect arbitrage relationship between money market interest rates and government bond interest rates does not hold, we should elaborate on why such a relationship does not hold in Japan.

Appendix 1 The Market Price of Risk

In a spot rate model, discount bonds are thought to be derivatives whose state variable is the instantaneous spot rate, and the process of the instantaneous spot rate is described by the following stochastic differential equation:

$$dr(t) = \mu(r(t), t)dt + \sigma(r(t), t)dz(t),$$

where $r(t)$ is an instantaneous spot rate, $z(t)$ is the Wiener process, and $\sigma(\cdot)$ is volatility. We assume that the price of the derivative with remaining maturity T follows the differentiable function $\xi(r(t), t)$. Applying Ito's lemma to the function enables us to write the derivation of $\xi(r(t), t)$ as

$$\frac{d\xi(r(t),t)}{\xi(r(t),t)} = \mu_\xi(t)dt + \sigma_\xi(t)dz(t) \qquad 0 \le t \le T, \tag{A-1}$$

$$\mu_\xi(t) = \frac{1}{\xi(t)}\left(\xi_r \mu + \xi_t + \frac{\xi_{rr}}{2}\sigma^2\right),$$

$$\sigma_\xi(t) = \frac{\xi_r \sigma}{\xi(t)}.$$

The argument of the duplication of a risk-free portfolio by two derivatives with different remaining maturity implies that the volatility-adjusted excess return of the portfolio does not depend on remaining maturity:

$$\frac{\mu_\xi(t) - r(t)}{\sigma_\xi(t)} \equiv \lambda(t). \tag{A-2}$$

The index $\lambda(t)$ is called the market price of risk, which shows the trade-off between risk and return on the derivative. In general, although the market price of risk follows the stochastic process that depends on past events, it becomes time-invariant in the Vasicek model under the assumption of no-arbitrage.

The market price of risk usually takes a negative value due to the following reasons: given the negative relationship between the instantaneous spot rate and the discount bond price, a rise in the instantaneous spot rate implies a negative value of ξ_r. This indicates that σ_ξ also takes a negative value since both σ and ξ are always positive. A no-arbitrage condition implies that the return of a risky asset exceeds that of a risk-free asset, that is, $\mu_{\xi}(t) > r(t)$ holds. The above argument enables us to know that the numerator is positive and the denominator is negative on the left side of equation (A-2). Thus, the market price of risk $\lambda(t)$ is always negative.

Appendix 2 Deriving Discount Bond Prices

Let us show the derivation process of equation (7). $v(t,T)$, price of a risk-free coupon bond at time t with maturity T, can be written as

$$v(t,T) = E^{P^*}\left[\exp\left\{-\int_t^T r_s^{\tau}\, ds\right\} | F_t\right].$$

Here, we can divide the right side of this equation into the following three parts:

$$E^{P^*}\left[1_{\{\tau < t\}} \exp\left\{-\int_t^T r_s^{\tau}\, ds\right\} | F_t\right] + E^{P^*}\left[1_{\{t \le \tau \le T\}} \exp\left\{-\int_t^T r_s^{\tau}\, ds\right\} | F_t\right] + E^{P^*}\left[1_{\{T < \tau\}} | F_t\right] \quad \text{(A-3)}$$

We attempt to calculate the three parts as follows:

First, the first part of equation (A-3) can be written as

$$E^{P^*}\left[1_{\{\tau < t\}} \exp\left\{-\int_t^T r_s^{\tau}\, ds\right\} | F_t\right] = 1_{\{\tau < t\}} E^{P^*}\left[\exp\left\{-\int_t^T r_s^{\tau}\, ds\right\} | F_t\right] = 1_{\{\tau < t\}} v^{\tau}(t,T), \quad \text{(A-4)}$$

where $v(t,T)$ is the bond price based on the Vasicek model in which an instantaneous spot rate at time $u(< t)$ is zero. $v(t,T)$ can be rewritten as

$$v^u(t,T) = H_1(T-t) e^{-H_2(T-t) r_t^u},$$

where H_1 and H_2 are equivalent to the definitions in equation (7).

Second, the third part of equation (A-3) can be written as[18]

$$
\begin{aligned}
& E^{P^*}\left[1_{\{T<\tau\}} \middle| F_t\right] \\
&= 1_{\{\tau<t\}} E^{P^*}\left[1_{\{T<t\}} \middle| \{\tau \le t\}\right] + 1_{\{t<\tau\}} E^{P^*}\left[1_{\{T<\tau\}} \middle| \{t<\tau\}\right] \\
&= 1_{\{t<\tau\}} P^*\left(T<\tau \middle| \{t<\tau\}\right) \\
&= 1_{\{t<\tau\}} \frac{P^*(T<\tau)}{P^*(t<\tau)} \\
&= 1_{\{t<\tau\}} \frac{1-\Psi(T)}{1-\Psi(t)}.
\end{aligned}
\tag{A-5}
$$

This equation can be derived from the property $1_{\{\tau<t\}} E^{P^*}\left[1_{\{T<t\}} \middle| \{\tau \le t\}\right] = 0$, which means that uncertainty before time τ is solely regarding the ending time of the ZIRP.

Third, to compute the second part of equation (A-3), we define the stochastic process, N_t, as

$$N_t \equiv 1_{\{\tau \le t\}}.$$

Also, the hazard rate of τ, l_t, is defined as

$$l_t \equiv \frac{\psi(t)}{1-\Psi(t)},$$

where Ψ and ψ are the probability distribution function and probability density function of τ, respectively. l_t is the deterministic function dependent on time. Using N_t and l_t, we define the following stchastic process as M_t.

$$M_t \equiv N_t - \int_0^t l_s (1 - N_s) ds \ . \tag{A-6}$$

M_t is known to become a martingale under P^*. The differential of equation (A-6) is written as

[18] See Marumo and Ieda (2000) and Lando (1998).

$$dN_t = dM_t + l_t(1 - N_t)dt\ , \qquad \text{(A-7)}$$

where Z_t is defined as follows:

$$Z_t \equiv v^t(t,T) = E^{P^*}\left[\exp\left\{-\int_t^T r_s^t ds\right\} \Big| F_t\right].$$

Like equation (A-4), this can be rewritten as

$$Z_t = v^t(t,T) = H_t(T-t)e^{-H_2(T-t)r_t^t}.$$

The definition of an instantaneous spot rate, $r_t^t = 0$, enables us to rewrite the above equation as

$$Z_t = H_1(T-t).$$

Finally, an expectation in the second part of equation (A-3) is written as

$$\begin{aligned} &E^{P^*}\left[1_{\{t\le\tau\le T\}} \exp\left\{-\int_t^T r_s^\tau ds\right\} \Big| F_t\right] \\ &= E^{P^*}\left[1_{\{t\le\tau\le T\}} E^{P^*}\left[\exp\left\{-\int_\tau^T r_s^\tau\right\} \Big| F_\tau\right] \Big| F_t\right] \\ &= E^{P^*}\left[1_{\{t\le\tau\le T\}} Z_\tau \ \Big| F_t\right]. \end{aligned}$$

From equation (A-7), this is rewritten as

$$1_{\{t\le\tau\le T\}} Z_\tau = \int_t^T Z_s dN_s = \int_t^T Z_s dM_s + \int_t^T Z_s l_s (1 - N_s) ds\ .$$

Furthermore, based on the property that the first part of this equation becomes a martingale, this expectation can be written as

$$\begin{aligned} &E^{P^*}\left[1_{\{t\le\tau\le T\}} Z_\tau \ \Big| F_t\right], \\ &= E^{P^*}\left[\int_t^T Z_s l_s (1 - N_s) ds \ \Big| F_t\right] \end{aligned} \qquad \text{(A-8)}$$

$$= \int_t^T Z_s l_s E^{P^*} \left[1 - N_s \quad \middle| F_t \right] ds$$
$$= \int_t^T Z_s l_s E^{P^*} \left[1_{\{s<\tau\}} \quad \middle| F_t \right] ds \quad .$$

Like equation (A-5), this is transformed into the following equation:

$$\begin{aligned} & E^{P^*} \left[1_{\{t \le \tau \le T\}} Z_\tau \quad \middle| F_t \right] \\ & = \int_t^T Z_s l_s 1_{\{t<\tau\}} \frac{1 - \Psi(s)}{1 - \Psi(t)} ds \\ & = 1_{\{t<\tau\}} \int_t^T v^s (s, T) \frac{\psi(s)}{1 - \Psi(s)} \frac{1 - \Psi(s)}{1 - \Psi(t)} ds \\ & = 1_{\{t<\tau\}} \int_t^T H_1 (T - s) \frac{\psi(s)}{1 - \Psi(t)} ds \quad . \end{aligned}$$

The first part of equation (A-3) is zero under $\tau > t$. Therefore, we can obtain equation (7) from equation (A-5) and the above equation.

Appendix 3 Estimation of Discount Bond Prices

In this appendix, we briefly explain the McCulloch model, one of the models to estimate discount bond prices from coupon bond prices. The yield curves on which our empirical analysis is based is calculated from the discount bond prices estimated using the McCulloch model. The following equation describes the relationship among coupon bond price P, accrued bond interest A, cash flow $C(t)$, and discount bond price $\delta(t)$. Here, t shows the remaining maturity of the discount bond price.

$$P + A = \sum_t C(t)\delta(t). \qquad \text{(A-9)}$$

Suppose that $1-\delta(t)$ can be written as the follwing linear combination of functions g_i,

$$\delta(t) = 1 + \sum_{i=1}^{k} a_i g_i(t),$$

where a_i is a parameter to be estimated. Since the McCulloch model assumes a three-dimensional cubic spline polynomial for $\delta(t)$, g_i should take the following form:

$$
g_i(t) = \begin{cases} 0 & (t < d_{i-1}) \\ \dfrac{(t - d_{i-1})^3}{6c_i} & (d_{i-1} \le t < d_i) \\ \dfrac{c_i^2}{6} + \dfrac{c_i(t - d_i)}{2} + \dfrac{(t - d_i)^2}{2} - \dfrac{(t - d_i)^3}{6(d_{i+1} - d_i)} & (d_i \le t < d_{i+1}) \\ (d_{i+1} - d_{i-1})\left[\dfrac{2d_{i+1} - d_i - d_{i-1}}{6} + \dfrac{t - d_{i+1}}{2}\right] & (d_{i+1} \le t), \end{cases}
$$

$$
g_k(t) = t,
$$
$$
c_i = d_i - d_{i-1},
$$
$$
i = 1,2,\cdots,k-1
$$

where d_i is the knot point. In our analysis, they are set per 1-year of remaining maturity from 1 to 19-year since we deal with coupon bonds with remaining maturity from 0 to 20-year. We estimate a_i employing the OLS method. Estimated a_i determines g_i and the yield curve is derived from discount bond prices in the form of the following equation:

$$
Y(t,T) = -\frac{\ln\left(1 + \sum_i a_i g_i(T - t)\right)}{T - t}.
$$

Appendix 4 Principal Component Analysis of Yield Curves

Principal component analysis is the statistical method of describing the variation in multivariate data in terms of a set of uncorrelated variables named principal components, each of which is a linear combination of the original variables. In general, some of the new variables account for most of the variation in the original data. Thus, new variables are often used to summarize the property of the data.

In this paper, we compare between the first component of principal component analysis and the estimated instantaneous spot rates estimated by the Vasicek and Fong model. This appendix shows the process of deriving the first component of principal component analysis from yield curves on a daily basis.

Letting i denote an indicator for the date, j for remaining maturity, and r_{ij} for interest rate with remaining maturity j at the i-th date, x_{ij} is defined as the difference in interest rates between the date and the previous date, that is,

$$x_{ij} = r_{ij} - r_{i-1,j}$$

Suppose u_i corresponds to the standardized vector of x_j. Mathematically, the vector u_j and matrix u_{ij} is defined by

$$u_j = \frac{x_j - \overline{x_j}}{s_j},$$
$$u_{ij} = [u_1, u_2, \cdots],$$

where λ_1 is the standard deviation. Let λ_1, $\lambda_2, \ldots$ in descending order denote the eigenvalues of the variance-covariance matrix R_{ij} of u_j [19]. Using $a(1)$, the eigenvector normalized to 1, which corresponds to λ_1, the first principal component $z(1)$ on the i-th date can be written as

$$z(1)_i = \sum_j a(1)_j \cdot u_{ij} .$$

It is generally known that the first principal component $z(1)_i$ represents the parallel shift of the yield curves. The first component needs to be accumulated for comparison with instantaneous spot rates since they are the principal component for the difference in interest rates: accumulation of $z(1)$ from the first date to the k-th date is denoted by $Z(1)_k$. We compute $Z(1)_k$ by the following equation with the value at the beginning date, September 1, 1994 equal to zero.

$$Z(1)_k = \sum_{i=1}^{k} z(1)_i .$$

[19] All eigenvalues of matrix R are positive because the variance-covariance matrix is proved to be a positive definite symmetric matrix.

The data we use is from September 1, 1994 to February 10, 1999.

Appendix 5 Estimation of Instantaneous Spot Rates

This appendix describes the Vasicek and Fong model for estimating instantaneous spot rates. This model assumes that the process of a discount bond price can be described by the exponential spline function. In estimating instantaneous spot rates, we do not define knot points in the model. It is because principal component factors other than a parallel shift of yield curves, such as a slope or a skewness factor, may affect estimated instantaneous spot rates if knot points are defined in the model. The discount bond prices implied by the Vasicek and Fong model without knot points are given by

$$v(t,T) = \theta_1 e^{-\omega(T-t)} + \theta_2 e^{-2\omega(T-t)} + \theta_3 e^{-3\omega(T-t)}.$$

Thus, each yield has the following form:

$$\begin{aligned} Y(t,T) &= -\frac{\ln v(t,T)}{T-t} \\ &= -\frac{\ln\left(\theta_1 e^{-\omega(T-t)} + \theta_2 e^{-2\omega(T-t)} + \theta_3 e^{-3\omega(T-t)}\right)}{T-t}. \end{aligned} \tag{A-10}$$

It should be noted that ω in equation (A-10) corresponds to the instantaneous forward rate in the distant future since the instantaneous forward rate is computed by

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T} \ln v(t,T) \\ &= \omega \frac{\theta_1 e^{-\omega(T-t)} + 2\theta_2 e^{-2\omega(T-t)} + 3\theta_3 e^{-3\omega(T-t)}}{\theta_1 e^{-\omega(T-t)} + \theta_2 e^{-2\omega(T-t)} + \theta_3 e^{-3\omega(T-t)}}, \end{aligned} \tag{A-11}$$

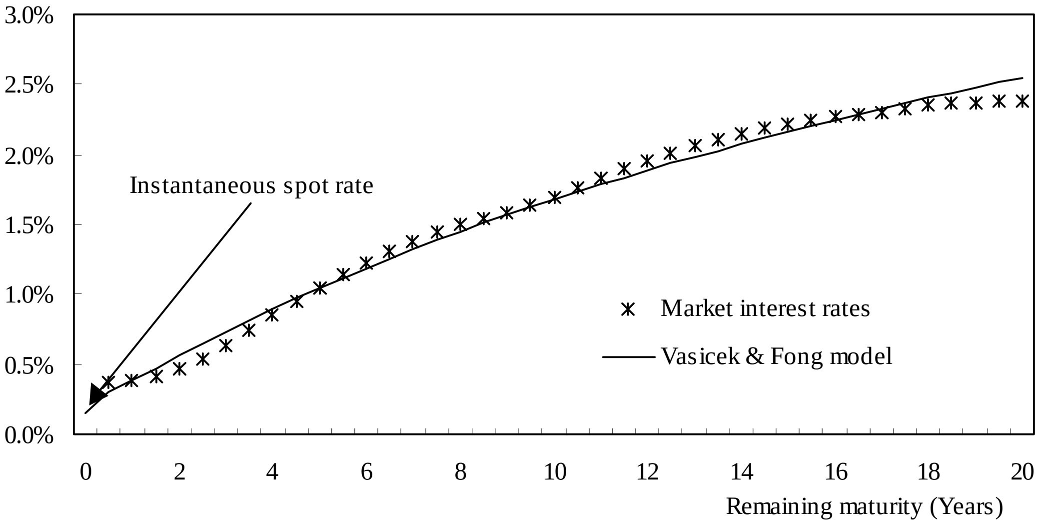
and the right side of equation (A-11) converges to ω when $T-t \to \infty$. The instantaneous forward rate in 20 years corresponds to ω in our analysis.

We estimate the parameters in equation (A-10) by the OLS method. The estimated parameters and ω yield the instantaneous spot rate using the following equation:

$$r(t) = f(t,t) = \frac{\omega(\theta_1 + 2\theta_2 + 3\theta_3)}{\theta_1 + \theta_2 + \theta_3}. \qquad \text{(A-12)}$$

Figure A-1 shows an example of the instantaneous spot rate and the yield curve estimated by the Vasicek and Fong model without knot points.

Figure A-1: Example of an Instantaneous Spot Rate Estimated by the Vasicek and Fong Model



Appendix 6　Comparative Analysis of Expectations on the Duration of the ZIRP Based on Various Probability Distribution Functions

We used the standard gamma distribution to describe the expectations of market participants on the duration of the ZIRP. It should be noted, however, that we do not have any particular rationale for choosing a particular type of probability distribution function. In this appendix, we try to compare the results derived from various probability distribution functions. To be more specific, we focus on such parameters as the mean of estimated probability density functions, the market price of risk, and the sum of residual squares. All probability distribution functions we use in this appendix meet the conditions necessary for the probability distribution functions to describe the expectations of market participants on the duration of the ZIRP. The specific conditions are unimodal, non-negativity, and parsimonious functional form. The estimation results are shown in table A-1 and A-2.

Table A-1: Estimation Results using Probability Distribution Functions with Two Parameters

	Lognormal distribution			Weibull distribution		
	E[τ]	λ	S R S	E[τ]	λ	S R S
07 Feb 00	6.107	-2.812	0.0006929	0.916	-2.348	0.0006154
07 Feb 00	7.774	-2.589	0.0005871	1.036	-2.027	0.0005083
04 Feb 00	2.592	-1.746	0.0004183	1.010	-1.671	0.0003693
25 Jan 00	7.320	-1.857	0.0005529	0.641	-1.557	0.0005029
22 Jan 00	5.078	-1.511	0.0003275	1.213	-1.338	0.0002552
16 Jan 00	4.677	-1.565	0.0002969	1.861	-1.483	0.0002043
08 Jan 00	4.206	-1.608	0.0003036	1.695	-1.535	0.0001980
16 Jan 00	4.309	-1.665	0.0003678	1.574	-1.568	0.0002578

	Gamma distribution			Standard gamma distribution		
	E[τ]	λ	S R S	E[τ]	λ	S R S
07 Feb 00	3.362	-2.454	0.0006093	2.378	-2.102	0.0010006
07 Feb 00	4.845	-3.091	0.0004820	2.881	-1.774	0.0008719
04 Feb 00	2.402	-1.820	0.0003764	1.925	-1.618	0.0004695
25 Jan 00	1.825	-0.762	0.0005196	1.537	-1.369	0.0008047
22 Jan 00	5.434	-3.674	0.0002090	3.123	-1.222	0.0004690
16 Jan 00	6.749	-3.841	0.0008843	4.146	-1.460	0.0002901
08 Jan 00	7.166	-4.605	0.0005657	3.701	-1.506	0.0002939
16 Jan 00	7.850	-5.987	0.0004432	3.581	-1.520	0.0003817

Table A-2: Estimation Results using Probability Distribution Functions with One Parameter

	Exponential distribution			Chi-square distribution		
	E[τ]	λ	S R S	E[τ]	λ	S R S
07 Feb 00	3.355	-2.387	0.0006125	2.782	-2.222	0.0007118
07 Feb 00	4.655	-2.214	0.0005437	3.350	-1.895	0.0006064
04 Feb 00	2.363	-1.717	0.0003926	2.259	-1.6953	0.0003742
25 Jan 00	1.846	-1.435	0.0006306	1.8233	-1.4275	0.0006014
22 Jan 00	5.320	-1.611	0.0004095	3.621	-1.3157	0.0002888
16 Jan 00	11.735	-3.085	0.0008659	4.771	-1.5981	0.0002981
08 Jan 00	7.901	-2.398	0.0007878	4.263	-1.6323	0.0002946
16 Jan 00	7.141	-2.276	0.0007332	4.123	-1.6421	0.0003178

	Standard Weibull distribution			Standard gamma distribution		
	E[τ]	λ	S R S	E[τ]	λ	S R S
07 Feb 00	0.980	-1.667	0.002823	2.378	-2.102	0.0010006
07 Feb 00	0.979	-1.266	0.002978	2.881	-1.774	0.0008719
04 Feb 00	0.980	-1.376	0.001621	1.925	-1.618	0.0004695
25 Jan 00	1.300	-1.293	0.001129	1.537	-1.369	0.0008047
22 Jan 00	0.980	-0.797	0.002394	3.123	-1.222	0.0004690
16 Jan 00	0.980	-0.772	0.003386	4.146	-1.460	0.0002901
08 Jan 00	0.980	-0.890	0.003220	3.701	-1.506	0.0002939
16 Jan 00	0.980	-0.924	0.003152	3.581	-1.520	0.0003817

(1) Probability distribution functions with two parameters

(a) Lognormal distribution

Probability density function:

$$\psi(x) = \frac{1}{\sqrt{2\pi}\,P_2 x}\exp\left[-\frac{(\log x - P_1)^2}{2P_2^2}\right]$$

Parameters: P_1, P_2

Mean:

$$E[x] = \exp\left(P_1 + \frac{1}{2}P_2^2\right)$$

The mean of the estimated probability distribution function rises in August 2000, which corresponds to the end of the ZIRP.

(b) Weibull distribution

Probability density function:

$$\psi(x) = \frac{P_4}{P_3}\left(\frac{x}{P_3}\right)^{P_4 - 1}\exp\left[-\left(\frac{x}{P_3}\right)^{P_4}\right] \qquad \text{(A-13)}$$

Parameters: P_3, P_4

Mean:

$$E[x] = P_3\Gamma\left(\frac{P_4 + 1}{P_4}\right)$$

The sum of residual squares remains small. The mean of the probability distribution functions, however, is lower than that of other probability distribution functions.

(c) Gamma distribution

Probability density function:

$$\psi(x) = \frac{x^{P_5 - 1}\exp\left(-\frac{x}{P_6}\right)}{P_6^{P_5}\Gamma(P_5)}$$

Parameters: P_5, P_6

Mean:

$$E[x] = P_5 P_6$$

The estimated market price of risk fluctuates widely. The probability distribution function with $P_6 = 1$ corresponds to the standard gamma distribution.

(2) Probability distribution functions with one parameter

(a) Exponential distribution

Probability density function:

$$\psi(x) = \frac{1}{P_7}\exp\left(-\frac{x}{P_7}\right)$$

Parameter: P_7

Mean:

$$E[x] = \frac{1}{P_7}$$

The estimated market price of risk is stable and the sum of residual squares remains small. This function does not seem to be realistic, however. It is because the mode of the probability density functions is always zero, which means the proportions of market participants who expect the ZIRP to end immediately is the largest in any situation.

(b) Chi-square distribution

Probability density function:

$$\psi(x) = \frac{1}{2^{P_8/2}\Gamma\left(\frac{P_8}{2}\right)}\exp\left(-\frac{x}{2}\right)x^{\frac{P_8}{2}-1}$$

Parameter: P_8

Mean:

$$E[x] = P_8$$

The estimation results are very similar to those of the standard gamma distribution.

(c) Standard Weibull distribution

Equation (A-13) is equivalent to the standard Weibull distribution when $P_3 = 1$. The sum of residual squares remains large and the mean of the probability density function moves unrealistically since it rises at the end of the ZIRP.

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