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Takushi Kurozumi*
takushi.kurozumi@boj.or.jp

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Bank of Japan
2-1-1 Nihonbashi Hongoku-cho, Chuo-ku, Tokyo 103-8660

* Research and Statistics Department

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Takushi Kurozumi†

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Abstract

Recent studies show that the Taylor rule possesses desirable properties in terms of generating determinacy and E-stability of rational expectations equilibria under sticky prices. This paper examines whether this policy rule retains these properties within a discrete-time money-in-utility-function model, employing three timings of money balances of the utility function that the existing literature contains: end-of-period timing and two types of cash-in-advance timing. This paper shows: (i) Even a small degree of non-separability of the utility function between consumption and real balances causes the Taylor rule to be much more likely to induce indeterminacy or E-instability if this rule responds not only to inflation but also to output or the output gap; (ii) Differences among the three timings strongly alter conditions for the Taylor rule to ensure both determinacy and E-stability.

JEL classification: E52

Keywords: Determinacy; E-stability; Monetary model; Taylor rule; Taylor principle

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†Corresponding author. Research and Statistics Department, Bank of Japan, 2-1-1 Nihonbashi Hongokucho, Chuo-ku, Tokyo 103-8660, Japan. Tel.: +81 3 3279 1111 ex. 3943; fax: +81 3 5255 6758. E-mail address: takushi.kurozumi@boj.or.jp

1 Introduction

Since Taylor's (1993) pioneering work, simple policy rules have received much attention in monetary policy analyses.[1] An important aspect of this recent research is to examine whether a proposed policy rule generates a (locally) determinate (i.e. unique non-explosive) equilibrium in a variety of rational expectations (RE) models. This line of research also discusses the relationship between determinacy of RE equilibria (REE) and the Taylor principle, which suggests that the nominal interest rate should be raised more than the increase in inflation.

Many recent analyses assume implicitly that if a policy rule generates a determinate REE, all agents can coordinate on that REE. In the face of this widespread belief, Bullard and Mitra (2002) address the questions of how and whether such coordination would arise by investigating expectational (or E-)stability of fundamental REE,[2] a concept emphasized recently by Evans and Honkapohja (1999, 2001). Roughly speaking, E-stability asks whether, for sufficiently small expectation errors of agents, a policy rule can lead temporary equilibria under such non-rational expectations to adjust over time toward the associated REE.[3] Thus, E-stability as well as determinacy of REE is a requirement policy rules must meet, as Bullard and Mitra (2002) stress. These authors use a sticky price model without money, which is prominent in recent monetary policy analyses,[4] and conclude that the Taylor rule in which the nominal interest rate responds to the current inflation rate and output gap possesses desirable properties in terms of generating both determinacy of REE and E-stability of fundamental REE.[5]

This paper examines whether the Taylor rule retains these properties within a discrete-time money-in-utility-function (MIUF) model with sticky prices. As McCallum and Nelson (1999) and Carlstrom and Fuerst (2001) indicate, the existing literature with discrete-time MIUF models contains three timings of money balances of the utility function: end-of-period (EOP)

[1]See Rotemberg and Woodford (1999), Benhabib, Schmitt-Grohé, and Uribe (2001), Carlstrom and Fuerst (2001, 2004), Bullard and Mitra (2002, 2003), Woodford (2003), among others.

[2]To distinguish two definitions of the minimal-state-variable (MSV) solution to linear RE models in the existing literature, which are given by McCallum (1983) and Evans and Honkapohja (1999, 2001), this paper refers to Evans and Honkapohja's MSV solutions as "fundamental".

[3]As Evans and Honkapohja (2001, Ch. 10) show for a broad class of linear stochastic models, if a fundamental REE is E-stable and non-explosive, it is least-squares learnable, i.e. stable under least-squares learning.

[4]See e.g. Rotemberg and Woodford (1999) and Woodford (2003, Ch. 4, Sec. 1).

[5]As McCallum (1999) argues, current values of inflation and the output gap are not available to actual central banks. Taking this into account, Bullard and Mitra suggest that rather than the one responding directly to them, the Taylor rule responding to current expectations about them is more practical, since these Taylor rules have the same properties in terms of generating both determinacy and E-stability.

timing and two types of cash-in-advance (CIA) timing.[6] Traditional literature starting from Brock (1974) has used EOP timing, which leads to the discrete-time analog to continuous-time MIUF models, while previous studies incorporating Clower's (1967) idea into MIUF models employ CIA timing, in which money balances held before consumption trading enter the utility function. With respect to timing of financial asset trading, CIA timing contains two approaches. One approach, whose idea is inspired by Lucas (1982) and Lucas and Stokey (1987), assumes financial asset trading in advance of consumption trading, so not only money balances held at the beginning of each period but also net gains from asset trading enter the utility function. A couple of papers by Carlstrom and Fuerst (2001, 2004) adopt this "Lucas-style" CIA timing. Another approach, which is based on Svensson's (1985) modification of Lucas' (1982) CIA constraint, assumes that financial asset trading follows consumption trading, which suggests that only beginning-of-period money balances enter the utility function. McCallum (1990), Woodford (1990), and McCallum and Nelson (1999), for instance, use this "Svensson-style" CIA timing.[7] The model with EOP or Svensson-style CIA timing can be considered a generalization of the model-without-money used in Bullard and Mitra (2002), because the former model takes the same form as the latter if the utility function is separable between consumption and real balances. In contrast, the model with Lucas-style CIA timing always differs from the model without money, as Carlstrom and Fuerst (2001) show. Thus, the goal of this paper is twofold. One goal is to examine implications of differences among these three timings for determinacy of REE and E-stability of fundamental REE. Another is to examine implications for them of non-separability of the utility function. For the case in which the Taylor rule responds only to inflation in continuous-time MIUF models, Benhabib, Schmitt-Grohé, and Uribe (2001) obtain the result that this non-separability has no implication for determinacy in the setting considered here (i.e. sticky prices, Ricardian fiscal policy, and production functions without money). As shown below, however, if the Taylor rule responds not only to inflation but also to output or the output gap, this non-separability has serious implications for both determinacy and E-stability.

In each case of timing this paper derives conditions under which the Taylor rule ensures

[6] As McCallum and Nelson indicate, this paper suggests that none of these three timings are fully "accurate" and each of them is actually just an approximation or a metaphor designed for transaction-facilitating services of the medium of exchange.

[7] Without financial asset trading, there is no difference between these two types of CIA timing. This happens in Lucas (2000).

determinacy of REE and E-stability of fundamental REE and illustrates them with reasonably calibrated parameter values. In the case of EOP timing, the condition for determinacy is consistent with that for E-stability. This condition can be interpreted as the long-run version of the Taylor principle: in the long run the nominal interest rate should be raised more than the increase in inflation, which is also indicated by Woodford (2003, Ch. 4, Sec. 2) and Bullard and Mitra (2002). Its implications here, however, differ notably from those in these studies with the model without money if the Taylor rule responds not only to inflation but also to output or the output gap.[8] In the model without money, the Taylor principle implies its long-run version (i.e. if the Taylor rule has an inflation coefficient greater than one, it generates both determinacy and E-stability) and hence the Taylor principle is a sufficient condition for both determinacy and E-stability. In the model here, once the extent of non-separability of the utility function exceeds a certain small threshold, the Taylor principle never implies its long-run version and it becomes a necessary condition, so corresponding to the output or output-gap coefficient, a larger inflation coefficient than the Taylor principle suggests is required for both determinacy and E-stability. In the case of Lucas-style CIA timing, the long-run version of the Taylor principle is the condition for E-stability, while another condition is also required for determinacy. Hence, the conditions for both determinacy and E-stability are more severe than those in the case of EOP timing. In the case of Svensson-style CIA timing, there are two sets of conditions for determinacy. Then, the long-run version of the Taylor principle is required for one set but not for another set, and the former set ensures E-stability while the latter set does not necessarily.[9]

In contract to the results of recent studies, this paper shows: (i) even a small degree of non-separability of the utility function between consumption and real balances causes the Taylor rule to be much more likely to induce indeterminacy or E-instability of REE if this rule responds not only to inflation but also to output or the output gap; (ii) the differences among

[8]If the Taylor rule responds only to inflation, the long-run version reduces to the usual Taylor principle.

[9]In this timing case, the model contains a lagged endogenous variable and thus we can consider two learning environments, both of which are studied by Evans and Honkapohja (2001, Sec. 10.3, 10.5). One environment allows agents to use current endogenous variables in expectation formation, while another does not. As Evans and Honkapohja indicate, the former environment induces a problem with simultaneous determination of current endogenous variables and expectations, which is critical to equilibria under non-rational expectations. Then, both the two sets of conditions for determinacy ensure E-stability in the former learning environment. In the latter environment, however, at least with the calibrated parameter values, one set that requires the long-run version of the Taylor principle ensures E-stability, while another set induces E-instability.

the three timings strongly alter the conditions for the Taylor rule to ensure both determinacy and E-stability.

The remainder of this paper is organized as follows. Section 2 presents a discrete-time sticky-price MIUF model with the Taylor rule in each case of timing. Section 3 derives conditions for this policy rule to ensure determinacy of REE and E-stability of fundamental REE and illustrates them with reasonably calibrated parameter values. Finally, Section 4 provides some concluding remarks.

2 Sticky price model with money

This section presents private agents' behavior, which can be derived from a discrete-time MIUF model with Calvo (1983) style staggered pricing of monopolistically competitive firms, and a monetary policy rule suggested by Taylor (1993). The existing literature with discrete-time MIUF models contains three timings of money balances of the utility function: EOP timing and two types of CIA timing.

2.1 End-of-period timing

Since Brock (1974), traditional literature with discrete-time MIUF models has employed EOP timing, which leads to the discrete-time analog to continuous-time MIUF models. This timing assumes that money balances held at the end of each period enter the utility function. For this timing case, Woodford (2003, Ch. 4, Sec. 3) derives optimal money-holding, spending and pricing behavior of private agents, represented by the following LM, IS and AS equations:

$$m_t = \eta_y Y_t - \eta_i i_t + v_t, \tag{1}$$

$$Y_t = \hat{E}_t Y_{t+1} - \sigma\{i_t - \hat{E}_t \pi_{t+1} + \chi(\hat{E}_t m_{t+1} - m_t)\} + g_t - \hat{E}_t g_{t+1}, \tag{2}$$

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \tilde{\kappa}(\omega + \sigma^{-1}) Y_t - \tilde{\kappa}\chi m_t - \tilde{\kappa}(\omega q_t + \sigma^{-1} g_t), \tag{3}$$

where Y_t, π_t, i_t are output, the inflation rate and the nominal interest rate in period t, and m_t is the real monetary base at the end of period t. Note that these variables denote log-deviations from steady state values and that $\hat{E}_t$ denotes a possibly non-rational expectations operator conditional on information available in period t. The exogenous disturbances, v_t, g_t, q_t, represent shocks relevant to money demand, consumption and production, respectively. In LM equation (1), $\eta_y, \eta_i > 0$ measure the output elasticity and interest rate semielasticity of money demand. In IS equation (2), $\sigma > 0$ measures the intertemporal elasticity of substitution in consumption and χ represents the degree of non-separability of the utility function between consumption and real balances. In AS equation (3), $\beta \in (0, 1)$ is the discount factor, $\tilde{\kappa} > 0$ represents the frequency of price adjustment,[10] and $\omega > 0$ measures the output elasticity of real marginal cost. Note that χ takes the same sign as the cross partial derivative of the utility function between consumption and real balances and that if the utility function is separable between these two arguments (i.e. $\chi = 0$), (2) and (3) take the same forms as IS and AS equations in the model without money, which is prominent in recent monetary policy analyses such as Rotemberg and Woodford (1999), Bullard and Mitra (2002), and Woodford (2003, Ch. 4, Sec. 1).

Monetary policy is assumed to be represented as an interest rate rule of Taylor's (1993) form

$$i_t \;=\; \phi_\pi \pi_t + \phi_y \left(Y_t - \gamma Y_t^n\right), \quad \phi_\pi, \phi_y \geq 0, \quad \gamma = 0, 1, \tag{4}$$

where Y_t^n denotes the natural rate of output, which would prevail in equilibrium under flexible prices, so $Y_t - Y_t^n$ represents the output gap. The Taylor rule (4) responds to output if $\gamma = 0$ and to the output gap if $\gamma = 1$. As in Woodford (2003, Ch. 4, Sec. 3), the natural rate of output here takes the form

$$Y_t^n \;=\; (\omega + \sigma^{-1} - \eta_y \chi)^{-1}(\omega q_t + \sigma^{-1} g_t + \chi v_t). \tag{5}$$

For simplicity, all the exogenous disturbances are assumed to follow univariate stationary first-order autoregressive processes

$$q_t \;=\; \rho_q q_{t-1} + \varepsilon_{q,t}, \quad g_t \;=\; \rho_g g_{t-1} + \varepsilon_{g,t}, \quad v_t \;=\; \rho_v v_{t-1} + \varepsilon_{v,t}, \tag{6}$$

where $\rho_j \in [0, 1)$, $j = q, g, v$, are autoregression parameters and $\varepsilon_{j,t}$, $j = q, g, v$, are white noises and may be correlated with each other.[11]

2.2 Lucas-style cash-in-advance timing

In contrast to the traditional literature, previous studies incorporating Clower's (1967) idea into MIUF models employ CIA timing, in which money balances held before consumption trading

[10] To be precise, $\tilde{\kappa}$ is a function of the frequency of price adjustment. See Footnote 15.

[11] In each case of timing, even if we alternatively assume that $|\,\rho_j\,| < 1$, $j = q, g, v$, conditions for determinacy are the same as those obtained below.

enter the utility function. With respect to timing of financial asset trading, CIA timing contains two approaches. One approach is based on the CIA constraint in Lucas (1982) and Lucas and Stokey (1987). These authors assume financial asset trading in advance of consumption trading. This suggests that not only money balances held at the beginning of each period but also net gains from financial asset trading enter the utility function, as Carlstrom and Fuerst (2001) illustrate in detail. Let a_t denote aggregate of such real balances that households hold before consumption trading in period t. Then, keeping the other basic structure of the model the same as in the case of EOP timing, we have LM, IS and AS equations of the forms[12]

$$\begin{aligned} a_t &= \eta_y Y_t - \eta_i \beta^{-1} i_t + v_t, && (7) \\ Y_t &= \hat{E}_t Y_{t+1} - \sigma\{\hat{E}_t i_{t+1} - \hat{E}_t \pi_{t+1} + \chi(\hat{E}_t a_{t+1} - a_t)\} + g_t - \hat{E}_t g_{t+1}, && (8) \\ \pi_t &= \beta \hat{E}_t \pi_{t+1} + \tilde{\kappa}(\omega + \sigma^{-1}) Y_t - \tilde{\kappa}\chi a_t - \tilde{\kappa}(\omega q_t + \sigma^{-1} g_t). && (9) \end{aligned}$$

The natural rate of output here is of the form (5). Note that the model here takes a very similar form to that with EOP timing, except that the nominal interest rate in IS equation (8) is scrolled forward one period from that in (2), as Carlstrom and Fuerst (2001) point out. Hence, when $\chi = 0$, AS equation (9) takes the same form as its counterpart in the model without money, while (8) does not.

2.3 Svensson-style cash-in-advance timing

Another approach to CIA timing is based on the CIA constraint of Svensson (1985), which is a modification of Lucas (1982). Svensson assumes consumption trading in advance of financial asset trading. This suggests that only beginning-of-period money balances, which equal money balances held at the end of the previous period, enter the utility function. This in turn leads to the next LM, IS and AS equations and natural rate of output:[13]

$$\begin{aligned} m_t &= \hat{E}_t \pi_{t+1} + \eta_y \hat{E}_t Y_{t+1} - \eta_i \beta^{-1} i_t + \hat{E}_t v_{t+1}, && (10) \\ Y_t &= \hat{E}_t Y_{t+1} - \sigma[i_t - \hat{E}_t \pi_{t+1} + \chi\{(m_t - \hat{E}_t \pi_{t+1}) - (m_{t-1} - \pi_t)\}] + g_t - \hat{E}_t g_{t+1}, && (11) \\ \pi_t &= \beta \hat{E}_t \pi_{t+1} + \tilde{\kappa}(\omega + \sigma^{-1}) Y_t - \tilde{\kappa}\chi(m_{t-1} - \pi_t) - \tilde{\kappa}(\omega q_t + \sigma^{-1} g_t), && (12) \\ Y_t^n &= (\omega + \sigma^{-1} - \eta_y \chi)^{-1}(\omega q_t + \sigma^{-1} g_t + \chi \hat{E}_{t-1} v_t). && (13) \end{aligned}$$

[12]See Carlstrom and Fuerst (2001) and Kurozumi (2004) for the derivation of these.
[13]See Kurozumi (2004) for the derivation of these.

Note that the lagged real monetary base m_{t-1} appears here. This is the key difference between Svensson-style CIA timing and the other two, because this monetary base is predetermined, which in turn affects conditions for determinacy of REE greatly as shown later. Note also that when $\chi = 0$, (11) and (12) take the same forms as the IS and AS equations in the model without money.

2.4 Main assumption and calibrated parameter values

To avoid complicated conditions for the Taylor rule (4) to ensure determinacy and E-stability for any value of χ, the ensuing analysis impose the following fairly reasonable assumption on χ.

Assumption 1 $0 \leq \chi < (\eta_y \sigma)^{-1}$

Woodford (2003, Ch. 4, Sec. 3) considers that the range $0 \leq \chi < \eta_y^{-1}(\omega + \sigma^{-1})$ is of greatest empirical relevance, because this range ensures that in the reduced AS equation with EOP timing that can be obtained by substituting (1) into (3) to eliminate the monetary base, the output elasticity of inflation is positive, i.e. $\kappa = \tilde{\kappa}(\omega + \sigma^{-1} - \eta_y \chi) > 0$ and the interest rate semielasticity of inflation is non-negative, i.e. $\kappa_i = \tilde{\kappa}\eta_i \chi \geq 0$. These also hold under Assumption 1, because $\omega > 0$ implies that the range of χ given by Assumption 1 is somewhat narrower than Woodford's.

The ensuing analysis also employs reasonably calibrated cases to illustrate conditions for determinacy and E-stability. The model contains nine parameters, $\beta, \tilde{\kappa}, \omega, \sigma, \eta_y, \eta_i, \chi, \phi_\pi, \phi_y$, for which values must be specified.[14] Parameter values in the baseline calibration are shown in Table 1. The first seven parameter values are chosen to be consistent with parameter values in Table 6.1 of Woodford (2003).[15] The coefficients ϕ_j, $j = \pi, y$, in the Taylor rule (4) are restricted within the range $0 \leq \phi_j \leq 3$, because with the other parameter values in Table 1, this range is wide enough to characterize whether the Taylor rule with non-negative coefficients guarantees determinacy and E-stability. To assess the robustness of the baseline calibration results with respect to the extent of non-separability of the utility function, χ, which takes a value of 0.02

[14] Basically, any specified parameter values for the exogenous disturbance processes (6) are not needed in the ensuing analysis.

[15] In Woodford the parameter $\tilde{\kappa}$ is given by $\tilde{\kappa} = (1-\alpha)(1-\alpha\beta)\{\alpha(1+\omega\theta)\}^{-1}$, where α and θ are the probability of not changing prices and the price elasticity of demand faced by individual firms.

in the baseline calibration, this paper chooses alternative values of $\chi = 0, 0.01, 0.03$.[16] Note that all the calibrated cases satisfy Assumption 1.

3 Determinacy and expectational stability of equilibrium

This section examines conditions under which the Taylor rule (4) ensures determinacy of REE and E-stability of fundamental REE in each case of timing.

3.1 End-of-period timing[17]

In the case of EOP timing, the economy's law of motion is given by (1)−(6). Under Assumption 1, using (1) and (4) to eliminate the monetary base and interest rate from (2) and (3) leads to a system of the form[18]

$$\hat{E}_t z_{t+1} = A z_t + B u_t, \quad z_t = [\pi_t \ Y_t]', \quad u_t = [q_t \ g_t \ v_t]', \tag{14}$$

$$A \equiv [A_{ij}] = \begin{bmatrix} \beta^{-1}(1-\kappa_i\phi_\pi) & -\beta^{-1}(\kappa+\kappa_i\phi_y) \\ \frac{\phi_\pi(1+\eta_i\chi)-A_{11}(1+\eta_i\chi\phi_\pi)}{\sigma^{-1}-\eta_y\chi+\eta_i\chi\phi_y} & 1+\frac{\phi_y-A_{12}(1+\eta_i\chi\phi_\pi)}{\sigma^{-1}-\eta_y\chi+\eta_i\chi\phi_y} \end{bmatrix}.$$

Recall that $\kappa = \tilde{\kappa}(\omega + \sigma^{-1} - \eta_y\chi) > 0$ and $\kappa_i = \tilde{\kappa}\eta_i\chi \geq 0$. From (6), we have $u_t = R_e u_{t-1} + \varepsilon_t$, where $R_e = \mathrm{diag}(\rho_q, \rho_g, \rho_v)$ and $\varepsilon_t = [\varepsilon_{q,t} \ \varepsilon_{g,t} \ \varepsilon_{v,t}]'$.

A REE with EOP timing is defined as a quartet of stochastic processes of inflation, output, the monetary base, and the interest rate such that a pair of the first two is a RE solution to system (14) and that the last two then follow from (1) and (4). Thus, such REE are determinate if and only if (14) has a determinate RE solution. Because inflation and output are non-predetermined, Proposition 1 of Blanchard and Kahn (1980) implies that (14) admits a determinate RE solution if and only if both eigenvalues of the matrix A are outside the unit circle.[19] Then, by Proposition C.1 of Woodford (2003), we have the next result.

[16]The choice of $\sigma = 6.4$ seems large in that it implies a risk aversion coefficient of $1/6.4 = 0.16$. To assess the robustness of the baseline calibration results, an alternative choice of $\sigma = 1$ is examined. The qualitative properties of the results survive with this choice, but of course, the quantitative ones differ. This claim also applies to the other parameter values.

[17]In the model with EOP timing, discretionary policy takes the same Taylor form as (4), so the analysis in this subsection can be also applied to this policy. See Kurozumi (2005) for details.

[18]The form of the matrix B is omitted since it is not needed in what follows.

[19]To be precise, this condition is sufficient for determinacy but only generically necessary. Throughout this paper, consideration of non-generic boundary cases is omitted.

Proposition 1 *Under Assumption 1, the Taylor rule* (4) *generates a determinate REE with EOP timing if and only if the following condition holds.*

$$\phi_\pi + \frac{1-\beta-\kappa_i}{\kappa}\phi_y \ > \ 1 \tag{15}$$

Proof. See Appendix A. ■

Condition (15) can be considered a generalization of (2.7) in Woodford (2003, Ch. 4, Sec. 2) or (20) in Bullard and Mitra (2002), each of which ensures determinacy of REE in the model without money, since (15) is consistent with each of them in the case of $\chi = 0$. As in these studies, (15) can be given the following economic interpretation. By (1)–(3) and (5), each percentage point of permanently higher inflation implies permanently higher output or output gap of $(1-\beta-\kappa_i)/\kappa$ percentage points. The left-hand side of condition (15) thus shows the long-run raise in the nominal interest rate by the Taylor rule (4) for each unit permanent increase in the inflation rate. Therefore, (15) can be interpreted as the long-run version of the Taylor principle: in the long run the nominal interest rate should be raised more than the increase in inflation, which is also pointed out by Woodford (2003, Ch. 4, Sec. 2) and Bullard and Mitra (2002). Its implications here, however, differ notably from those in these studies with the model without money, as discussed later. Note that when the Taylor rule (4) responds only to inflation, i.e. $\phi_y = 0$, the long-run version (15) reduces to the usual Taylor principle, i.e. $\phi_\pi > 1$, regardless of non-separability of the utility function between real balances and consumption, and thus the non-separability has no implication for determinacy of the REE with EOP timing. This is consistent with Proposition 6 of Benhabib et al. (2001), which shows with continuous-time MIUF models that the Taylor principle (i.e. $\rho'(\pi^*) > 1$ in their terms) guarantees determinacy of equilibria under sticky prices, regardless of the non-separability, when fiscal policy is Ricardian and money balances never enter production functions, which is the setting considered here. One point of this paper is, however, that if the Taylor rule (4) responds not only to inflation but also to output or the output gap, even a small degree of the non-separability leads this policy rule to be much more likely to induce indeterminacy and E-instability of REE, as shown later.

We next consider E-stability of fundamental REE with EOP timing, following Evans and

Honkapohja (2001, Sec. 10.3). The fundamental RE solution to (14) is then given by

$$z_t = \bar{k}_e + \bar{\Gamma}_e u_t, \quad \bar{k}_e = 0, \quad \mathrm{vec}(\bar{\Gamma}_e) = \{(R_e \otimes I) - (I \otimes A)\}^{-1}\mathrm{vec}(B), \tag{16}$$

where I denotes a conformable identity matrix throughout the paper. For the study of learning, all agents are assumed to be endowed with a perceived law of motion (PLM) of z_t, $z_t = k_e + \Gamma_e u_t$, which corresponds to the fundamental RE solution (16).[20] Because $\hat{E}_t z_{t+1} = k_e + \Gamma_e R_e u_t$, substituting this into (14) leads to an actual law of motion (ALM) of z_t, $z_t = A^{-1}k_e + A^{-1}(\Gamma_e R_e - B)u_t$, where the matrix A is invertible under Assumption 1. We can then define a mapping T from the PLM to the ALM as $T(k_e, \Gamma_e) = [A^{-1}k_e, A^{-1}(\Gamma_e R_e - B)]$. For the fundamental RE solution $(\bar{k}_e, \bar{\Gamma}_e)$ in (16) to be E-stable, the matrix differential equation $\frac{d}{d\tau}(k_e, \Gamma_e) = T(k_e, \Gamma_e) - (k_e, \Gamma_e)$ must have local asymptotic stability at the solution. This is the case if and only if all eigenvalues of the matrix $(A^{-1} - I)$ have negative real parts, because $DT_k(\bar{k}_e) = A^{-1}$, $DT_\Gamma(\bar{k}_e, \bar{\Gamma}_e) = R_e \otimes A^{-1}$, and $R_e = \mathrm{diag}(\rho_q, \rho_g, \rho_v)$ with $\rho_j \in [0, 1)$, $j = q, g, v$. Then, by the Routh-Hurwitz theorem,[21] we have the next result.

Proposition 2 *Under Assumption 1, the necessary and sufficient condition for the Taylor rule* (4) *to guarantee E-stability of the fundamental REE* (16) *is the same as* (15).

Proof. See Appendix B. ■

Condition (15) can be also considered a generalization of (24) in Bullard and Mitra (2002), which ensures E-stability of fundamental REE in the model without money. Proposition 1 and 2 in conjunction show the following result.

Corollary 1 *Under Assumption 1, the next three are equivalent: (i) The Taylor rule* (4) *generates a determinate REE with EOP timing, (ii)* (4) *ensures E-stability of the fundamental REE* (16)*, and (iii)* (4) *satisfies the long-run version of the Taylor principle* (15).

We now use the calibrated cases to illustrate condition (15) for determinacy and E-stability of the REE with EOP timing. Figure 1 shows regions of coefficients on the Taylor rule (4) ensuring determinacy and E-stability for each value of χ. From (15), we have the next boundary

[20] Throughout the paper, constant terms are included in the PLM. This is because the model is a log-linear approximation around a steady state, so all agents may misunderstand the steady state.

[21] See e.g. Samuelson (1947).

between the region generating a determinate E-stable REE and that inducing indeterminacy and E-instability.

$$b_1(\phi_\pi, \phi_y) \;\equiv\; \phi_\pi + \frac{1-\beta-\kappa_i}{\kappa}\phi_y - 1 \;=\; 0 \tag{17}$$

Note that (17) always has an intercept of one in the axis of the inflation coefficient ϕ_π. In the case of $\chi = 0$, where the model here takes the same form as the model without money, (17) has not only the ϕ_π-intercept of one but also a positive intercept of $\kappa/(1-\beta)$ in the axis of the output or output-gap coefficient ϕ_y, as can be seen in Figure 1(i).[22] Thus, as Bullard and Mitra (2002) and Woodford (2003, Ch. 4, Sec. 2) point out, the Taylor principle (i.e. $\phi_\pi > 1$) implies its long-run version (15) and hence is a sufficient condition for both determinacy and E-stability. However, once χ exceeds $\chi^* \equiv (1-\beta)(\tilde{\kappa}\eta_i)^{-1} = 0.0095$, which yields $1-\beta-\kappa_i < 0$, (17) can no longer have a positive ϕ_y-intercept, as can be seen in Figure 1(ii)-(iv). This in turn suggests that if the Taylor rule (4) responds not only to inflation but also to output or the output gap, the Taylor principle never implies its long-run version (15) and it becomes a necessary condition for both determinacy and E-stability. This is in stark contrast to the implications of the long-run version of the Taylor principle in previous studies with the model without money.

3.2 Lucas-style cash-in-advance timing

In the case of Lucas-style CIA timing, if $\sigma^{-1} - \eta_y\chi - (1-\eta_i\chi\beta^{-1})\phi_y \neq 0$, then combining (4), (5) and (7)–(9) yields a system of the form[23]

$$\hat{E}_t z_{t+1} \;=\; C z_t + D u_t, \quad z_t \;=\; [\pi_t \;\, Y_t]', \quad u_t \;=\; [q_t \;\, g_t \;\, v_t]', \tag{18}$$

$$C \;\equiv\; [C_{ij}] \;=\; \begin{bmatrix} \beta^{-1}(1-\kappa_i\phi_\pi\beta^{-1}) & -\beta^{-1}(\kappa+\kappa_i\phi_y\beta^{-1}) \\ \frac{\eta_i\chi\phi_\pi\beta^{-1} - C_{11}\{1-\phi_\pi(1-\eta_i\chi\beta^{-1})\}}{\sigma^{-1}-\eta_y\chi-(1-\eta_i\chi\beta^{-1})\phi_y} & 1 + \frac{\phi_y - C_{12}\{1-\phi_\pi(1-\eta_i\chi\beta^{-1})\}}{\sigma^{-1}-\eta_y\chi-(1-\eta_i\chi\beta^{-1})\phi_y} \end{bmatrix}.$$

From (6), we have $u_t = R_\ell u_{t-1} + \varepsilon_t$, where $R_\ell = \mathrm{diag}(\rho_q, \rho_g, \rho_v)$ and $\varepsilon_t = [\varepsilon_{q,t} \;\, \varepsilon_{g,t} \;\, \varepsilon_{v,t}]'$.

By an argument similar to that in the case of EOP timing, the REE with Lucas-style CIA timing are determinate if and only if system (18) has a determinate RE solution, that is, both eigenvalues of the matrix C are outside the unit circle. The next result thus follows.

[22] This figure is the same as Figure 1 in Bullard and Mitra (2002), since the same parameter values are used.

[23] The form of the matrix D is omitted, since it is not needed in what follows.

Proposition 3 *Under Assumption 1 and* $\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y \neq 0$*, the Taylor rule* (4) *leads to a determinate REE with Lucas-style CIA timing if and only if the next two conditions hold.*[24]

$$\phi_\pi + \frac{1 - \beta - \kappa_i\beta^{-1}}{\kappa}\phi_y \ > \ 1 \tag{19}$$

$$\frac{\kappa - 2\kappa_i\omega\beta^{-1}}{\kappa + 2(1+\beta)(\sigma^{-1} - \eta_y\chi)}\phi_\pi + \frac{(1+\beta)(1 - 2\eta_i\chi\beta^{-1}) - \kappa_i\beta^{-1}}{\kappa + 2(1+\beta)(\sigma^{-1} - \eta_y\chi)}\phi_y \ < \ 1 \tag{20}$$

Proof. See Appendix C. ■

From an argument similar to that in the case of EOP timing, (19) can be also interpreted as the long-run version of the Taylor principle. For determinacy of the REE here, another condition (20) is required as well as the long-run version of the Taylor principle (19). Hence the Taylor rule (4) is more likely to induce indeterminacy of the REE here than in the case of EOP timing.[25]

We next investigate E-stability of fundamental REE with Lucas-style CIA timing. The fundamental RE solution to (18) is given by

$$z_t \ = \ \bar{k}_\ell + \bar{\Gamma}_\ell u_t, \quad \bar{k}_\ell \ = \ 0, \quad \text{vec}(\bar{\Gamma}_\ell) \ = \ \{(R_\ell \otimes I) - (I \otimes C)\}^{-1}\text{vec}(D). \tag{21}$$

By an argument analogous to that in the case of EOP timing, this fundamental RE solution is E-stable if and only if all eigenvalues of the matrix $(C^{-1} - I)$ have negative real parts, where the matrix C is invertible under Assumption 1. The next results thus follow.

Proposition 4 *Under Assumption 1 and* $\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y \neq 0$*, the necessary and sufficient condition for the Taylor rule* (4) *to guarantee E-stability of the fundamental REE* (21) *is the same as* (19).[26]

Proof. See Appendix D. ■

[24]If $\chi = 0$, the difference between the model here and the model without money is that in the former model the expected interest rate appears in the IS equation while in the latter the current one does, as Carlstrom and Fuerst (2001) indicate. Hence, in this case (19) and (20) are consistent with (40) and (41) in Bullard and Mitra (2002), which examines under what condition a Taylor rule responding to the expected inflation rate and output gap leads to determinacy of REE in the model without money. Note that (39) in Bullard and Mitra is redundant because it is implied by (40) and (41). When $\chi > 0$, however, there is no such a relationship.

[25]The long-run version of the Taylor principle here, (19), differs slightly from the one (15) in the case of EOP timing, and (19) has a somewhat more severe policy implication than (15) because $\beta < 1$.

[26]If $\chi = 0$, the same reason as that mentioned in Footnote 24 ensures that (19) is consistent with (42) in Bullard and Mitra (2002) under which the Taylor rule responding to the expected inflation rate and output gap ensures E-stability of fundamental REE in the model without money.

Corollary 2 *If the Taylor rule (4) brings about a determinate REE with Lucas-style CIA timing, this REE is E-stable.*

With the calibrated cases we now illustrate the conditions obtained above. Figure 2 shows regions of coefficients on the Taylor rule (4) ensuring determinacy of REE and E-stability of the fundamental REE (21) for each value of χ. Note first that (20) holds for any value of $\chi \geq \chi^{***} \equiv (\omega + \sigma^{-1})(\eta_y + 2\omega\eta_i\beta^{-1})^{-1} = 0.023$ with the other calibrated parameter values, because the coefficients on $\phi_\pi, \phi_y \geq 0$ in (20) are non-positive.[27] This suggests that in the case of $\chi \geq \chi^{***}$, the only relevant condition for both determinacy and E-stability is the long-run version of the Taylor principle (19), which has the same serious implications as the one (15) in the case of EOP timing. In the cases of $\chi = 0, 0.01$, (19) and (20) yield the following two boundaries.

$$b_2(\phi_\pi, \phi_y) \equiv \phi_\pi + \frac{1 - \beta - \kappa_i\beta^{-1}}{\kappa}\phi_y - 1 = 0 \tag{22}$$

$$b_3(\phi_\pi, \phi_y) \equiv \frac{\kappa - 2\kappa_i\omega\beta^{-1}}{\kappa + 2(1+\beta)(\sigma^{-1} - \eta_y\chi)}\phi_\pi + \frac{(1+\beta)(1 - 2\eta_i\chi\beta^{-1}) - \kappa_i\beta^{-1}}{\kappa + 2(1+\beta)(\sigma^{-1} - \eta_y\chi)}\phi_y - 1 = 0 \tag{23}$$

If $b_2(\phi_\pi, \phi_y) > 0$ and $b_3(\phi_\pi, \phi_y) < 0$, which corresponds to the range that is the right-hand side of (22) and below (23) in Figure 2(i)-(ii), the pair (ϕ_π, ϕ_y) of coefficients on (4) satisfies (19) and (20) and hence ensures both determinacy and E-stability.[28] Otherwise, the Taylor rule (4) induces indeterminacy of the REE here. These features remain as long as $0 \leq \chi \leq \chi^{**} \equiv \beta(1+\beta)[\eta_i\{\tilde{\kappa} + 2(1+\beta)\}]^{-1} = 0.018$, which yields $(1+\beta)(1 - 2\eta_i\chi\beta^{-1}) - \kappa_i\beta^{-1} \geq 0$, although (22) turns right around the ϕ_π-intercept of one and (23) goes up. Once $\chi > \chi^{**}$, the range ensuring both determinacy and E-stability changes drastically, since (23) has a negative ϕ_y-intercept, which implies that the range given by $b_2(\phi_\pi, \phi_y) > 0$ and $b_3(\phi_\pi, \phi_y) < 0$ becomes the right-hand side of (22) and above (23). In the case of $\chi = 0.02$, (23) has the ϕ_π-intercept of 48, so if the pair (ϕ_π, ϕ_y) satisfies the long-run version of the Taylor principle (19) and the inflation coefficient ϕ_π is less than 48, it generates a determinate E-stable REE. In the case of $\chi = 0.03$, because $\chi \geq \chi^{***} = 0.023$, (23) disappears from the range of non-negative coefficients on (4), and hence (19) completely features the region yielding both determinacy and E-stability.

[27] Note that $\chi \geq \chi^{***} \equiv (\omega + \sigma^{-1})(\eta_y + 2\omega\eta_i\beta^{-1})^{-1} \Leftrightarrow \kappa - 2\kappa_i\omega\beta^{-1} \leq 0$ and that $\kappa - 2\kappa_i\omega\beta^{-1} \leq 0$ implies $(1+\beta)(1 - 2\eta_i\chi\beta^{-1}) - \kappa_i\beta^{-1} < 0$.

[28] For the same reasons as mentioned in Footnote 22 and 24, Figure 2(i) is the same as Figure 3 in Bullard and Mitra (2002).

3.3 Svensson-style cash-in-advance timing

In the case of Svensson-style CIA timing, combining (4), (6) and (10)−(13) yields, under Assumption 1, a system of the form[29]

$$\hat{E}_t z_{t+1} \;=\; F z_t + G u_t, \quad z_t \;=\; [\pi_t \; Y_t \; m_{t-1}]', \quad u_t \;=\; [q_t \; g_t \; v_t \; \varepsilon_{v,t}]', \tag{24}$$

$$F \;\equiv\; [F_{ij}] \;=\; \left[\begin{array}{ccc} \beta^{-1}(1-\tilde{\kappa}\chi) & -\tilde{\kappa}\beta^{-1}(\omega+\sigma^{-1}) & \tilde{\kappa}\chi\beta^{-1} \\ \frac{\chi+\phi_\pi(1-\eta_i\chi\beta^{-1})-F_{11}}{\sigma^{-1}-\eta_y\chi} & \frac{\sigma^{-1}+\phi_y(1-\eta_i\chi\beta^{-1})-F_{12}}{\sigma^{-1}-\eta_y\chi} & -\frac{\chi+F_{13}}{\sigma^{-1}-\eta_y\chi} \\ F_{11}+\eta_y F_{21}-\eta_i\phi_\pi\beta^{-1} & F_{12}+\eta_y F_{22}-\eta_i\phi_y\beta^{-1} & F_{13}+\eta_y F_{23} \end{array}\right].$$

From (6), we have $u_t = R_s u_{t-1} + \varepsilon_t$, where $R_s = \mathrm{diag}(\rho_q, \rho_g, \rho_v, 0)$ and $\varepsilon_t = [\varepsilon_{q,t} \;\; \varepsilon_{g,t} \;\; \varepsilon_{v,t} \;\; \varepsilon_{v,t}]'$.

A REE with Svensson-style CIA timing is defined as a quartet of stochastic processes of inflation, output, the monetary base, and the interest rate such that a triplet of the first three is a RE solution to system (24) and that the last one then follows from (4). Thus, such REE are determinate if and only if (24) has a determinate RE solution. By Blanchard and Kahn's (1980) Proposition 1, this is the case if and only if exactly one eigenvalue of the matrix F is inside the unit circle and the other two are outside the unit circle, since the lagged monetary base is predetermined but inflation and output are not. Then, by Proposition C.2 of Woodford (2003), we have the next result.

Proposition 5 *Under Assumption 1, the Taylor rule* (4) *generates a determinate REE with Svensson-style CIA timing if and only if either (Case I)* (19) *and the next condition hold*

$$\frac{\kappa - 2\kappa_i\omega\beta^{-1}}{\kappa + 2(1+\beta)(\sigma^{-1}-\eta_y\chi)}\phi_\pi + \frac{(1+\beta)(1-2\eta_i\chi\beta^{-1}) - \kappa_i\beta^{-1}}{\kappa + 2(1+\beta)(\sigma^{-1}-\eta_y\chi)}\phi_y \;>\; -1; \tag{25}$$

or (Case II) the two strict inequalities opposite to (19) *and* (25) *hold.*

Proof. See Appendix E. ■

As noted above, (19) can be considered a generalization of (2.7) in Woodford (2003, Ch. 4, Sec. 2) or (20) in Bullard and Mitra (2002) and can be also interpreted as the long-run version of the Taylor principle.[30] One point of Proposition 5 is that the long-run version of the Taylor principle is not always required for determinacy of the REE here. As discussed later, even if its coefficients do not satisfy (19), the Taylor rule (4) may generate a determinate REE. This is in stark contrast to the results in the other two timing cases.

[29]The form of the matrix G is omitted, since it is not needed in what follows.

[30]If $\chi = 0$, (25) holds and hence the only relevant condition is (19).

We next examine E-stability of fundamental REE with Svensson-style CIA timing. Because system (24) contains a lagged endogenous variable, which is the lagged monetary base, we can consider two learning environments, both of which are studied by Evans and Honkapohja (2001, Sec. 10.3, 10.5). One environment allows agents to use current endogenous variables in expectation formation, while another does not. As Evans and Honkapohja indicate, the former environment induces a problem with simultaneous determination of current endogenous variables and expectations, which is critical to equilibria under non-rational expectations.

We begin with the learning environment in which current endogenous variables are available in expectation formation. Partitioning system (24) conformably with current and lagged endogenous variables leads to[31]

$$\hat{E}_t \tilde{z}_{t+1} \;=\; F_{zz}\tilde{z}_t + F_{zm} m_{t-1} + G_z u_t, \tag{26}$$

$$m_t \;=\; F_{mz}\tilde{z}_t + F_{mm} m_{t-1} + G_m u_t, \tag{27}$$

$$\tilde{z}_t = \begin{bmatrix} \pi_t \\ Y_t \end{bmatrix}, \quad F_{zz} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad F_{zm} = \begin{bmatrix} F_{13} \\ F_{23} \end{bmatrix}, \quad F_{mz} = [F_{31}\;\; F_{32}], \quad F_{mm} = F_{33}.$$

Then, fundamental RE solutions to (24) or to (26)−(27) are given by

$$\tilde{z}_t \;=\; \bar{k}_z + \bar{\Phi}_z m_{t-1} + \bar{\Gamma}_z u_t, \tag{28}$$

$$m_t \;=\; \bar{k}_m + \bar{\Phi}_m m_{t-1} + \bar{\Gamma}_m u_t, \tag{29}$$

$$\bar{k}_z = 0_{2\times 1}, \quad \bar{k}_m = 0, \quad \bar{\Phi}_z(F_{mz}\bar{\Phi}_z + F_{mm}) = F_{zz}\bar{\Phi}_z + F_{zm}, \quad \bar{\Phi}_m = F_{mz}\bar{\Phi}_z + F_{mm},$$

$$\mathrm{vec}(\bar{\Gamma}_z) = [\{I \otimes (F_{zz} - \bar{\Phi}_z F_{mz})\} - (R_s \otimes I)]^{-1}\mathrm{vec}(\bar{\Phi}_z G_m - G_z), \quad \bar{\Gamma}_m = F_{mz}\bar{\Gamma}_z + G_m.$$

Note that $\bar{\Phi}_m$, $\bar{\Gamma}_z$, and $\bar{\Gamma}_m$ are uniquely determined given a $\bar{\Phi}_z$ but that $\bar{\Phi}_z$ is not uniquely determined. Corresponding to (28), all agents are assumed to have a PLM of $\tilde{z}_t$,

$$\tilde{z}_t \;=\; k_z + \Phi_z m_{t-1} + \Gamma_z u_t. \tag{30}$$

Under the learning environment considered here, from (27) and (30) we have

$$\hat{E}_t \tilde{z}_{t+1} \;=\; k_z + \Phi_z F_{mz}\tilde{z}_t + \Phi_z F_{mm} m_{t-1} + (\Phi_z G_m + \Gamma_z R_s) u_t.$$

Substituting this into (26) leads to an ALM of z_t,

$$\tilde{z}_t \;=\; \Psi(\Phi_z)\{k_z + (\Phi_z F_{mm} - F_{zm}) m_{t-1} + (\Phi_z G_m + \Gamma_z R_s - G_z) u_t\} \tag{31}$$

[31] The forms of the matrices G_z and G_m are omitted, since they are not needed in what follows.

provided the matrix $\Psi(\Phi_z) \equiv (F_{zz} - \Phi_z F_{mz})^{-1}$ exists.[32] We can then define a mapping T from the PLM (30) to the ALM (31) as

$$T(k_z, \Phi_z, \Gamma_z) \; = \; [\Psi(\Phi_z)k_z, \Psi(\Phi_z)(\Phi_z F_{mm} - F_{zm}), \Psi(\Phi_z)(\Phi_z G_m + \Gamma_z R_s - G_z)].$$

For the fundamental RE solutions (28)−(29) to be E-stable, the matrix differential equation $\frac{d}{d\tau}(k_z, \Phi_z, \Gamma_z) = T(k_z, \Phi_z, \Gamma_z) - (k_z, \Phi_z, \Gamma_z)$ must have local asymptotic stability at $(\bar{k}_z, \bar{\Phi}_z, \bar{\Gamma}_z)$. This is the case if and only if all eigenvalues of three matrices, $\{\Psi(\bar{\Phi}_z) - I\}$, $\{(F_{mz}\bar{\Phi}_z + F_{mm})\Psi(\bar{\Phi}_z) - I\}$, and $[\{R_s \otimes \Psi(\bar{\Phi}_z)\} - I]$, have negative real parts. Because $\bar{\Phi}_m = F_{mz}\bar{\Phi}_z + F_{mm}$ and $R_s = \text{diag}(\rho_q, \rho_g, \rho_v, 0)$ with $\rho_j \in [0, 1)$, $j = q, g, v$, we have the next result.

Lemma 1 *Suppose that current endogenous variables are available in expectation formation. Then, the Taylor rule* (4) *guarantees E-stability of the fundamental REE* (28)−(29) *if and only if real parts of all eigenvalues of two matrices,* $\Psi(\bar{\Phi}_z)$ *and* $\bar{\Phi}_m\Psi(\bar{\Phi}_z)$*, are less than one provided that* $\Psi(\bar{\Phi}_z) \equiv (F_{zz} - \bar{\Phi}_z F_{mz})^{-1}$ *exists.*

Lemma 1 provides no explicit condition for E-stability, so it might seem impossible to find relationships between determinacy and E-stability like those obtained in the other two timing cases. Following McCallum (2004), however, we can obtain the next result.[33]

Proposition 6 *Suppose that all agents can use current endogenous variables in expectation formation. Then, if the Taylor rule* (4) *generates a determinate REE with Svensson-style CIA timing, this REE is E-stable.*

Proof. See Appendix F. ■

We turn next to another learning environment in which current endogenous variables are not available in expectation formation. Then, the PLM (30), together with (27), yields

$$\hat{E}_t \tilde{z}_{t+1} \; = \; (I + \Phi_z F_{mz})k_z + \Phi_z(F_{mz}\Phi_z + F_{mm})m_{t-1} + \{\Phi_z(F_{mz}\Gamma_z + G_m) + \Gamma_z R_s\}u_t.$$

[32] This matrix exists at least when the REE here are determinate.

[33] For a broad class of linear stochastic models of the form given by Evans and Honkapohja (2001, Sec. 10.3), McCallum (2004) adopts the undetermined coefficient method to show E-stability of the determinate REE under learning with current endogenous variables. See also Kurozumi and McCallum (2005), which employs Klein's (2000) method to demonstrate the same result as in McCallum (2004) for a general class of linear stochastic models of the form given by McCallum (1998).

Substituting this into (26) leads to an ALM of z_t,

$$\begin{aligned}\tilde{z}_t &= F_{zz}^{-1}(I + \Phi_z F_{mz})k_z + F_{zz}^{-1}\{\Phi_z(F_{mz}\Phi_z + F_{mm}) - F_{mm}\}m_{t-1} \\ &\quad + F_{zz}^{-1}[\{\Phi_z(F_{mz}\Gamma_z + G_m) + \Gamma_z R_s\} - G_z]u_t \end{aligned} \tag{32}$$

provided the matrix F_{zz} is invertible. Then, a mapping T from the PLM (30) to the ALM (32) can be defined as

$$T\begin{pmatrix} k_z \\ \Phi_z \\ \Gamma_z \end{pmatrix}' = \begin{pmatrix} F_{zz}^{-1}(I + \Phi_z F_{mz})k_z \\ F_{zz}^{-1}\{\Phi_z(F_{mz}\Phi_z + F_{mm}) - F_{zm}\} \\ F_{zz}^{-1}[\{\Phi_z(F_{mz}\Gamma_z + G_m) + \Gamma_z R_s\} - G_z] \end{pmatrix}'.$$

For the fundamental RE solutions (28)−(29) to be E-stable, real parts of all eigenvalues of three matrices, $\{F_{zz}^{-1}(I + \bar{\Phi}_z F_{mz}) - I\}$, $[F_{zz}^{-1}\{(F_{mz}\bar{\Phi}_z + F_{mm})I + \bar{\Phi}_z F_{mz}\} - I]$, and $[(R_s \otimes F_{zz}^{-1}) + \{I \otimes (F_{zz}^{-1}\bar{\Phi}_z F_{mz})\} - I]$, are negative. Because $\bar{\Phi}_m = F_{mz}\bar{\Phi}_z + F_{mm}$, we have the next result.

Lemma 2 *Suppose that F_{zz} is invertible and that all agents cannot use current endogenous variables in expectation formation. Then the Taylor rule* (4) *guarantees E-stability of the fundamental REE* (28)−(29) *if and only if real parts of all eigenvalues of three matrices, $F_{zz}^{-1}(I + \bar{\Phi}_z F_{mz})$, $F_{zz}^{-1}(\bar{\Phi}_m I + \bar{\Phi}_z F_{mz})$, and $[(R_s \otimes F_{zz}^{-1}) + \{I \otimes (F_{zz}^{-1}\bar{\Phi}_z F_{mz})\}]$, are less than one.*

In the learning environment considered here, there seems no clear relationship between determinacy and E-stability.

We now use the calibrated cases to illustrate the conditions obtained above. For each value of χ, Figure 3 shows regions of coefficients on the Taylor rule (4) ensuring determinacy of REE and E-stability of the fundamental REE (28)−(29) in the two learning environments.[34] Note that (25) holds for any value of $0 \le \chi \le \chi^{**} = 0.018$ with the other calibrated parameter values, since the coefficients on $\phi_\pi, \phi_y \ge 0$ in (25) are non-negative. This implies that in the cases of $\chi = 0, 0.01$, the only relevant condition for determinacy is the long-run version of the Taylor principle (19), which leads to the boundary (22) in Figure 3A(i) and 3B(i). In the case of $\chi = 0$, where the model here takes the same form as the model without money, the region yielding E-stability is consistent with that yielding determinacy, as claimed by Corollary 1 in the case

[34] As in Lemma 2, E-stability under learning without current endogenous variables requires values of the matrix $R_s = \mathrm{diag}(\rho_q, \rho_g, \rho_v, 0)$. With the other calibrated parameter values, any values of $\rho_j \in [0, 1)$, $j = q, g, v$, provide the same results shown in Figure 3.

of EOP timing. When $\chi = 0.01$, the coefficient region ensuring determinacy still guarantees E-stability in both the two learning environments, but the region of indeterminacy induces two non-explosive fundamental REE. Then, these two fundamental REE are E-unstable if agents are learning without current endogenous variables (See Figure 3B(i)); but if agents are learning with these variables (See Figure 3A(i)), the fundamental REE associated with eigenvalues of the matrix F in (24) in order of decreasing modulus is E-unstable while another fundamental REE is E-stable. Following McCallum (2004), this paper refers to the former fundamental REE as "MOD". In the cases of $\chi = 0.02, 0.03$, from (25) there emerges another boundary between the region generating determinacy and that inducing indeterminacy

$$b_4(\phi_\pi, \phi_y) \equiv \frac{\kappa - 2\kappa_i \omega \beta^{-1}}{\kappa + 2(1+\beta)(\sigma^{-1} - \eta_y \chi)} \phi_\pi + \frac{(1+\beta)(1 - 2\eta_i \chi \beta^{-1}) - \kappa_i \beta^{-1}}{\kappa + 2(1+\beta)(\sigma^{-1} - \eta_y \chi)} \phi_y + 1 = 0. \quad (33)$$

Proposition 5 then leads to two regions of coefficients on (4) ensuring determinacy. One region is given by $b_2(\phi_\pi, \phi_y) > 0$ and $b_4(\phi_\pi, \phi_y) > 0$, which corresponds to the range below (33) within the right-hand side of (22) in Figure 3A(ii)-(iii) and 3B(ii)-(iii). This region satisfies the long-run version of the Taylor principle (19). Another is given by $b_2(\phi_\pi, \phi_y) < 0$ and $b_4(\phi_\pi, \phi_y) < 0$, which corresponds to the range above (33) within the left-hand side of (22). This region never satisfies such Taylor principle (19). Then, if agents are learning with current endogenous variables (See Figure 3A(ii)-(iii)), the region of determinacy guarantees E-stability as Proposition 6 claims, and the region of indeterminacy contains two non-explosive fundamental REE and consists of two sub-regions: in one region labeled as "Indeterminate A", only the MOD fundamental REE is E-stable; while in another region labeled as "Indeterminate B", only the non-MOD fundamental REE is E-stable. In contrast, if agents are learning without current endogenous variables (See Figure 3B(ii)-(iii)), one of two regions ensuring determinacy that never meets the long-run version of the Taylor principle (19) induces a determinate, but E-unstable, REE; and another region ensures both determinacy and E-stability but shrinks within the range of non-negative coefficients on (4) as χ increases.[35] In the region of indeterminacy there are two non-explosive fundamental REE both of which are E-unstable.

[35]This region disappears from the range of non-negative coefficients on (4) if $\chi > 0.12$.

4 Concluding remarks

In this paper we have examined determinacy and E-stability of REE in the discrete-time sticky-price MIUF model with the Taylor rule, employing three timing of money balances of the utility function that the existing literature contains. In contrast to the results of recent studies, this paper has shown that even a small degree of non-separability of the utility function between consumption and real balances causes the Taylor rule to be much more likely to induce indeterminacy or E-instability of REE if this rule responds not only to inflation but also to output or the output gap. Such a problem might be mitigated by endowing the Taylor rule with interest rate smoothing, as Bullard and Mitra (2003) suggest for backward-looking and forward-looking Taylor rules in the model without money. This paper has also demonstrated that the differences among the three timings of money balances strongly alter the conditions for the Taylor rule to ensure both determinacy and E-stability. A companion paper by Kurozumi (2004) investigates implications of these differences for optimal policy and shows that such differences have notable effects on optimal policy under discretion. Many recent studies of monetary policy have used the model without money. In the face of this widespread use of it, this paper suggests that we employ the associated model with money to assess the robustness of results obtained with the model without money.

Appendix

A Proof of Proposition 1

Proposition C.1 of Woodford (2003) implies that the necessary and sufficient condition for both eigenvalues of the matrix A to be outside the unit circle is either

$$\text{(i)}\quad \det A > 1, \quad \det A - \operatorname{tr} A > -1, \quad \det A + \operatorname{tr} A > -1; \quad \text{or}$$

$$\text{(ii)}\quad \det A - \operatorname{tr} A < -1, \quad \det A + \operatorname{tr} A < -1,$$

where

$$\begin{aligned}
\det A - 1 &= \frac{(\kappa + \kappa_i \omega)\phi_\pi + \phi_y}{\sigma^{-1} - \eta_y \chi + \eta_i \chi \phi_y} + 1 - \beta, \\
\det A - \operatorname{tr} A + 1 &= \frac{\kappa \phi_\pi + (1 - \beta - \kappa_i)\phi_y - \kappa}{\sigma^{-1} - \eta_y \chi + \eta_i \chi \phi_y}, \\
\det A + \operatorname{tr} A + 1 &= \frac{(\kappa + 2\kappa_i \omega)\phi_\pi + (1 + \beta + \kappa_i)\phi_y + \kappa}{\sigma^{-1} - \eta_y \chi + \eta_i \chi \phi_y} + 2(1 + \beta).
\end{aligned}$$

Under Assumption 1 we have that $\det A - 1 > 0$ and $\det A + \operatorname{tr} A + 1 > 0$. This implies that only (i) is relevant and the necessary and sufficient condition for the Taylor rule (4) to generate a determinate REE with EOP timing is $\det A - \operatorname{tr} A > -1$, which can be reduced to (15).

B Proof of Proposition 2

The characteristic equation of the matrix $(A^{-1} - I)$ can be written as

$$\lambda^2 + a\lambda + b = 0,$$

where

$$\begin{aligned}
a &= \frac{(2\kappa + \kappa_i \omega)\phi_\pi + \{2 - \beta - \kappa_i + \eta_i \chi (1 - \beta)\}\phi_y + (1 - \beta)(\sigma^{-1} - \eta_y \chi) - \kappa}{(\kappa + \kappa_i \omega)\phi_\pi + (1 + \eta_i \chi)\phi_y + \sigma^{-1} - \eta_y \chi}, \\
b &= \frac{\kappa \phi_\pi + (1 - \beta - \kappa_i)\phi_y - \kappa}{(\kappa + \kappa_i \omega)\phi_\pi + (1 + \eta_i \chi)\phi_y + \sigma^{-1} - \eta_y \chi}.
\end{aligned}$$

Note that $(\kappa + \kappa_i \omega)\phi_\pi + (1 + \eta_i \chi)\phi_y + \sigma^{-1} - \eta_y \chi > 0$ under Assumption 1. The Routh-Hurwitz theorem implies that the necessary and sufficient condition for all eigenvalues of the matrix $(A^{-1} - I)$ to have negative real parts is that $a > 0$ and $b > 0$. Then we have

$$a = b + \frac{(\kappa + \kappa_i \omega)\phi_\pi + \{1 + \eta_i \chi (1 - \beta)\}\phi_y + (1 - \beta)(\sigma^{-1} - \eta_y \chi)}{(\kappa + \kappa_i \omega)\phi_\pi + (1 + \eta_i \chi)\phi_y + \sigma^{-1} - \eta_y \chi}.$$

Thus, under Assumption 1, $b > 0$ implies $a > 0$ and hence the necessary and sufficient condition for the Taylor rule (4) to guarantee E-stability of the fundamental REE (16) is $b > 0$, which can be reduced to (15).

C Proof of Proposition 3

Suppose that $\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y \neq 0$. Proposition C.1 of Woodford (2003) implies that the necessary and sufficient condition for both eigenvalues of the matrix C to be outside the unit circle is either

$$\begin{aligned} &\text{(i)} \quad \det C > 1,\ \det C - \operatorname{tr} C > -1,\ \det C + \operatorname{tr} C > -1;\ \text{ or} \\ &\text{(ii)} \quad \det C - \operatorname{tr} C < -1,\ \det C + \operatorname{tr} C < -1, \end{aligned}$$

where

$$\begin{aligned} \det C - 1 &= \frac{\kappa_i\omega\beta^{-1}\phi_\pi + \phi_y}{\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y} + 1 - \beta, \\ \det C - \operatorname{tr} C + 1 &= \frac{\kappa\phi_\pi + (1 - \beta - \kappa_i\beta^{-1})\phi_y - \kappa}{\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y}, \\ \det C + \operatorname{tr} C + 1 &= \frac{(2\kappa_i\omega\beta^{-1} - \kappa)\phi_\pi + (1 + \beta + \kappa_i\beta^{-1})\phi_y + \kappa}{\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y} + 2(1 + \beta). \end{aligned}$$

First consider the case in which $\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y < 0$. Under Assumption 1 we have

$$\det C - 1 = \frac{\kappa_i\omega\beta^{-1}\phi_\pi + \{\beta + \eta_i\chi\beta^{-1}(1 - \beta)\}\phi_y + (1 - \beta)(\sigma^{-1} - \eta_y\chi)}{\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y} < 0.$$

Hence only (ii) is relevant and the necessary and sufficient condition for the Taylor rule (4) to generate a determinate REE with Lucas-style CIA timing is that $\det C - \operatorname{tr} C + 1 < 0$ and $\det C + \operatorname{tr} C + 1 < 0$, which can be reduced to (19) and (20), respectively.

Next consider the case in which $\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y > 0$. Suppose that (ii) holds, i.e., $\det C - \operatorname{tr} C + 1 < 0$, $\det C + \operatorname{tr} C + 1 < 0$. Then $\det C + \operatorname{tr} C + 1 < 0$ implies

$$(2\kappa_i\omega\beta^{-1} - \kappa)\phi_\pi + (1 + \beta + \kappa_i\beta^{-1})\phi_y + \kappa + 2(1 + \beta)\{\sigma^{-1} - \eta_y\chi - (1 - \eta_i\chi\beta^{-1})\phi_y\} < 0,$$

so we have

$$(2\kappa_i\omega\beta^{-1} - \kappa)\phi_\pi + (1 + \beta + \kappa_i\beta^{-1})\phi_y + \kappa < 0.$$

From $\det C - \operatorname{tr} C + 1 < 0$ we have

$$\kappa\phi_\pi + (1 - \beta - \kappa_i\beta^{-1})\phi_y - \kappa < 0$$

and hence

$$\begin{aligned}
&(2\kappa_i\omega\beta^{-1} - \kappa)\phi_\pi + (1 + \beta + \kappa_i\beta^{-1})\phi_y + \kappa \\
&= 2(\kappa_i\omega\beta^{-1}\phi_\pi + \phi_y) - \{\kappa\phi_\pi + (1 - \beta - \kappa_i\beta^{-1})\phi_y - \kappa\} \\
&> 0,
\end{aligned}$$

which is a contradiction. Thus, only (i) is relevant. Because $\det C - 1 > 0$, the necessary and sufficient condition for the Taylor rule (4) to generate a determinate REE with Lucas-style CIA timing is that $\det C - \operatorname{tr} C + 1 > 0$ and $\det C + \operatorname{tr} C + 1 > 0$, which can be reduced to (19) and (20), respectively.

D Proof of Proposition 4

The characteristic equation of the matrix $(C^{-1} - I)$ can be written as

$$\lambda^2 + c\lambda + d = 0,$$

where

$$\begin{aligned}
c &= \frac{(\kappa + \kappa_i\omega\beta^{-1})\phi_\pi + \{1 - \kappa_i\beta^{-1} + \eta_i\chi(\beta^{-1} - 1)\}\phi_y + (1 - \beta)(\sigma^{-1} - \eta_y\chi) - \kappa}{\kappa_i\omega\beta^{-1}\phi_\pi + \eta_i\chi\beta^{-1}\phi_y + \sigma^{-1} - \eta_y\chi}, \\
d &= \frac{\kappa\phi_\pi + (1 - \beta - \kappa_i\beta^{-1})\phi_y - \kappa}{\kappa_i\omega\beta^{-1}\phi_\pi + \eta_i\chi\beta^{-1}\phi_y + \sigma^{-1} - \eta_y\chi}.
\end{aligned}$$

Note that $\kappa_i\omega\beta^{-1}\phi_\pi + \eta_i\chi\beta^{-1}\phi_y + \sigma^{-1} - \eta_y\chi > 0$ under Assumption 1. The Routh-Hurwitz theorem implies that the necessary and sufficient condition for all eigenvalues of the matrix $(C^{-1} - I)$ to have negative real parts is that $c > 0$ and $d > 0$. Then we have

$$c = d + \frac{\kappa_i\omega\beta^{-1}\phi_\pi + \{\beta + \eta_i\chi(\beta^{-1} - 1)\}\phi_y + (1 - \beta)(\sigma^{-1} - \eta_y\chi)}{\kappa_i\omega\beta^{-1}\phi_\pi + \eta_i\chi\beta^{-1}\phi_y + \sigma^{-1} - \eta_y\chi}.$$

Thus, under Assumption 1, $d > 0$ implies $c > 0$ and hence the necessary and sufficient condition for the Taylor rule (4) to guarantee E-stability of the fundamental REE (21) is $d > 0$, which can be reduced to (19).

E Proof of Proposition 5

The characteristic equation of the matrix F can be written as

$$\lambda^3 + e_2\lambda^2 + e_1\lambda + e_0 = 0,$$

where

$$\begin{aligned}
e_2 &= -\beta^{-1} - 1 - \frac{\kappa\beta^{-1} + (1 - \eta_i\chi\beta^{-1})\phi_y}{\sigma^{-1} - \eta_y\chi}, \\
e_1 &= \beta^{-1} + \beta^{-1}\frac{(\kappa - \kappa_i\omega\beta^{-1})\phi_\pi + \{1 - \kappa_i\beta^{-1} - \eta_i\chi(1+\beta^{-1})\}\phi_y}{\sigma^{-1} - \eta_y\chi}, \\
e_0 &= \beta^{-2}\frac{\kappa_i\omega\phi_\pi + \eta_i\chi\phi_y}{\sigma^{-1} - \eta_y\chi}.
\end{aligned}$$

Proposition C.2 of Woodford (2003) implies that exactly one eigenvalue of the matrix F is inside the unit circle and the other two are outside the unit circle if and only if one of the following three is satisfied.

(i) $1 + e_2 + e_1 + e_0 < 0,\ -1 + e_2 - e_1 + e_0 > 0;$

(ii) $1 + e_2 + e_1 + e_0 > 0,\ -1 + e_2 - e_1 + e_0 < 0,\ e_0^2 - e_0e_2 + e_1 - 1 > 0;$

(iii) $1 + e_2 + e_1 + e_0 > 0,\ -1 + e_2 - e_1 + e_0 < 0,\ e_0^2 - e_0e_2 + e_1 - 1 < 0,\ |e_2| > 3.$

Under Assumption 1 we have

$$\begin{aligned}
e_0^2 - e_0e_2 + e_1 - 1 &= \frac{\beta^{-2}(\kappa_i\omega\phi_\pi + \eta_i\chi\phi_y)[\kappa_i\omega\beta^{-2}\phi_\pi + \{1 + \eta_i\chi\beta^{-2}(1-\beta)\}\phi_y]}{(\sigma^{-1} - \eta_y\chi)^2} \\
&\quad + \frac{\beta^{-1}\{\kappa\kappa_i\omega\beta^{-2} + (\kappa + \kappa_i\omega\beta^{-2})(\sigma^{-1} - \eta_y\chi)\}\phi_\pi}{(\sigma^{-1} - \eta_y\chi)^2} \\
&\quad + \frac{\beta^{-1}[\kappa_i\omega\beta^{-2} + \{1 + \kappa_i\beta^{-2}(1-\beta) + \eta_i\chi(\beta^{-2} - 1)\}(\sigma^{-1} - \eta_y\chi)]\phi_y}{(\sigma^{-1} - \eta_y\chi)^2} \\
&> 0.
\end{aligned}$$

Hence (iii) is not the case and the necessary and sufficient condition for the Taylor rule (4) to generate a determinate REE with Svensson-style CIA timing is either (Case I)

$$1 + e_2 + e_1 + e_0 > 0, \quad -1 + e_2 - e_1 + e_0 < 0,$$

which can be reduced to (19) and (25); or (Case II)

$$1 + e_2 + e_1 + e_0 < 0, \quad -1 + e_2 - e_1 + e_0 > 0.$$

F Proof of Proposition 6

For the matrix F in (24), the Schur decomposition theorem[36] ensures the existence of a unitary matrix Z and a lower triangular matrix T such that

$$Z^H F Z \; = \; T, \tag{34}$$

where Z^H denotes the Hermitian transpose of Z. Note that the diagonal elements $\{t_{ii}\}$ of the triangular matrix T are eigenvalues of the matrix F. Without contradicting the foregoing theorem, these eigenvalues t_{ii} (and associated columns of Z) can be arranged in order of decreasing modulus, so unstable eigenvalues (i.e. eigenvalues with moduli greater than one[37]) come first in T. Because system (24) has a determinate RE solution, it follows by Theorem 4.1 of Klein (2000) that $Z_{mm} \neq 0$, T_{zz} is invertible and the evolution of $\tilde{z}_t'$ and m_t is given by

$$\tilde{z}_t \; = \; \tilde{\Phi}_z m_{t-1} + \tilde{\Gamma}_z u_t, \quad \tilde{\Phi}_z \; = \; Z_{zm} Z_{mm}^{-1}, \tag{35}$$

$$m_t \; = \; \tilde{\Phi}_m m_{t-1} + \tilde{\Gamma}_m u_t, \quad \tilde{\Phi}_m \; = \; T_{mm}, \tag{36}$$

where the matrices Z_{ij} and T_{ij} are obtained by partitioning Z and T conformably with $\tilde{z}_t'$ and m_t such that

$$Z \; = \; \begin{bmatrix} Z_{zz} & Z_{zm} \\ Z_{mz} & Z_{mm} \end{bmatrix}, \quad T \; = \; \begin{bmatrix} T_{zz} & 0_{2\times 1} \\ T_{mz} & T_{mm} \end{bmatrix}.$$

Because $Z^{-1} = Z^H$ and $Z_{mm} \neq 0$, we have that Z_{zz}^H is invertible and $Z_{zm} Z_{mm}^{-1} = -(Z_{zz}^H)^{-1} Z_{zm}^H$, where each Z_{ij}^H is constructed in the same way as Z_{ij}. Then, from (35) we have

$$\tilde{\Phi}_z \; = \; -(Z_{zz}^H)^{-1} Z_{zm}^H. \tag{37}$$

From the upper left block in (34), we have $Z_{zz}^H F_{zz} + Z_{zm}^H F_{mz} = T_{zz} Z_{zz}^H$ and hence

$$F_{zz} + (Z_{zz}^H)^{-1} Z_{zm}^H F_{mz} \; = \; (Z_{zz}^H)^{-1} T_{zz} Z_{zz}^H, \tag{38}$$

since Z_{zz}^H is invertible. From (37) and (38) we have

$$F_{zz} - \tilde{\Phi}_z F_{mz} \; = \; (Z_{zz}^H)^{-1} T_{zz} Z_{zz}^H. \tag{39}$$

Note that $\Psi(\tilde{\Phi}_z) = (F_{zz} - \tilde{\Phi}_z F_{mz})^{-1}$ exists, since T_{zz} and Z_{zz}^H are invertible. Then, (39) implies that $\Psi(\tilde{\Phi}_z)$ has the same eigenvalues as T_{zz}^{-1}. By determinacy of RE solutions to (24), these

[36]See e.g. Golub and Van Loan (1996).

[37]Recall that consideration of non-generic boundary cases is omitted throughout this paper.

eigenvalues are the inverses of unstable eigenvalues of the matrix F, so all eigenvalues of $\Psi(\tilde{\Phi}_z)$ are inside the unit circle. Also, the determinacy implies that T_{mm} is the stable eigenvalue of the matrix F and hence from (36) we have $|\,\tilde{\Phi}_m\,| = |\,T_{mm}\,| < 1$. Therefore, all eigenvalues of $\Psi(\tilde{\Phi}_z)$ and $\tilde{\Phi}_m\Psi(\tilde{\Phi}_z)$ have real parts less than one and thus E-stability of the determinate REE follows from Lemma 1.

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Table 1:
Baseline calibration, quarterly

β	discount factor	0.99
$\tilde{\kappa}$	frequency of price adjustment	0.038
ω	output elasticity of real marginal cost	0.47
σ	intertemporal elasticity of substitution in consumption	6.4
η_y	output elasticity of money demand	1
η_i	interest rate semielasticity of money demand	28
χ	degree of non-separability of utility function between consumption and real balances	0.02
ϕ_π	inflation coefficient in Taylor rule (4)	$0 \leq \phi_\pi \leq 3$
ϕ_π	output or output-gap coefficient in Taylor rule (4)	$0 \leq \phi_\pi \leq 3$

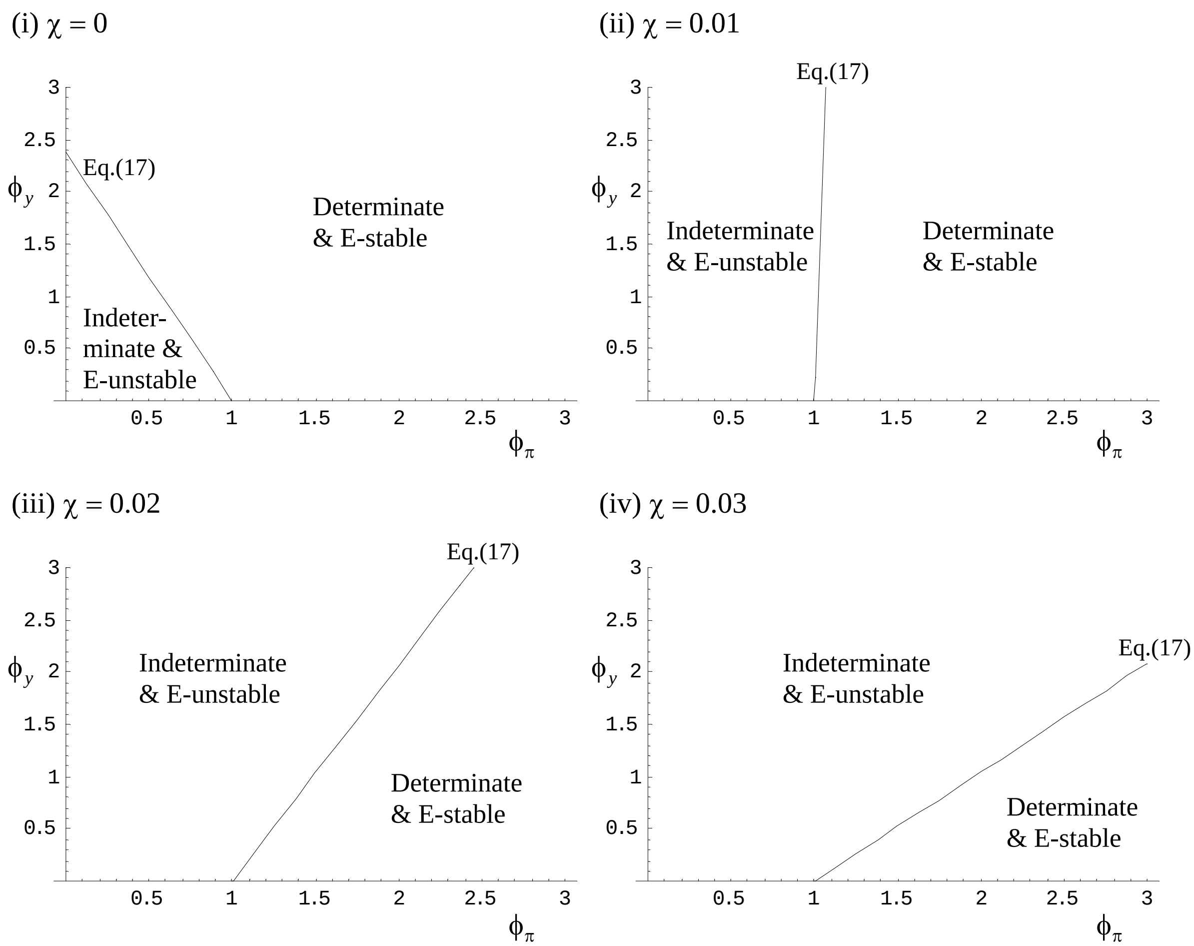

Figure 1. Regions of coefficients on Taylor rule ensuring determinacy and E-stability of REE with EOP timing.

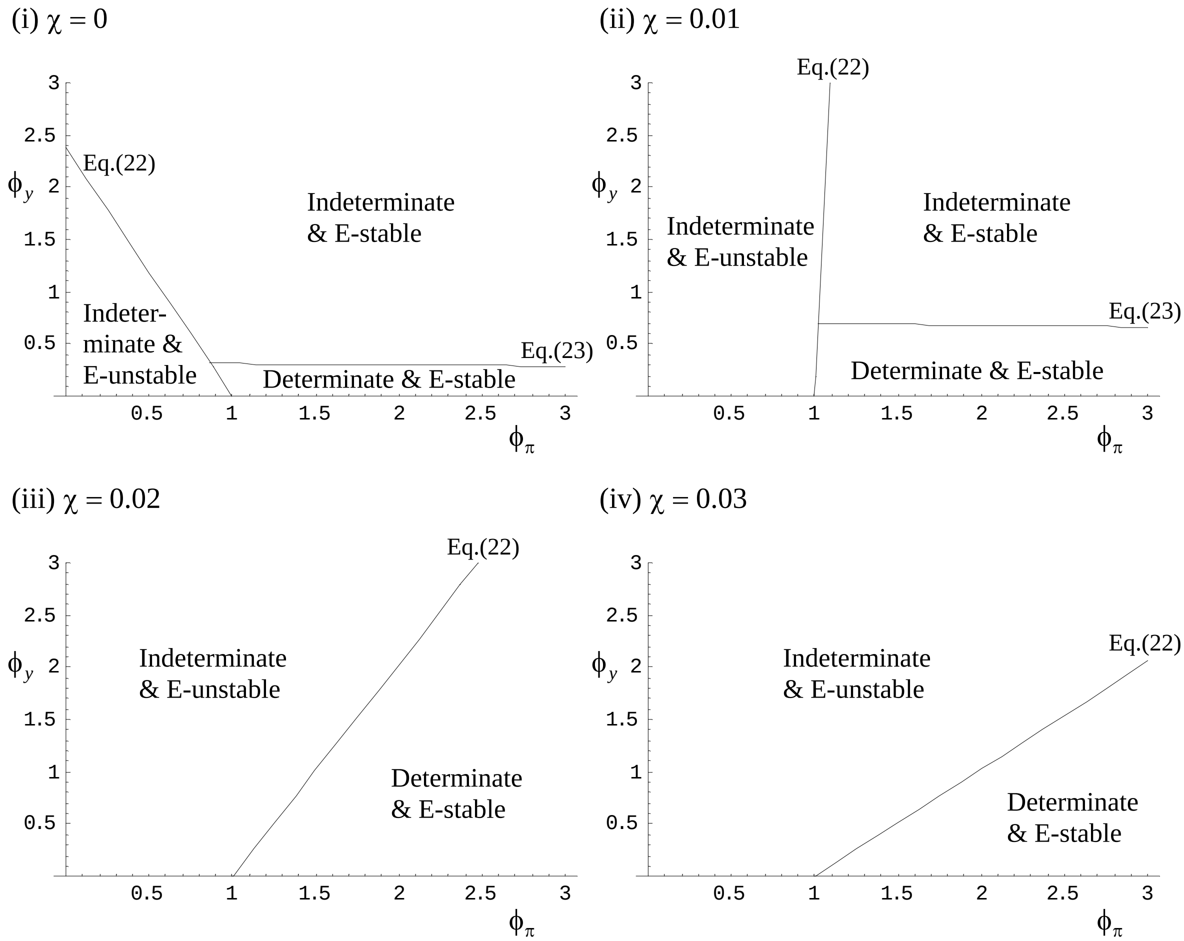

Figure 2. Regions of coefficients on Taylor rule ensuring determinacy of REE and E-stability of fundamental REE with Lucas-style CIA timing.

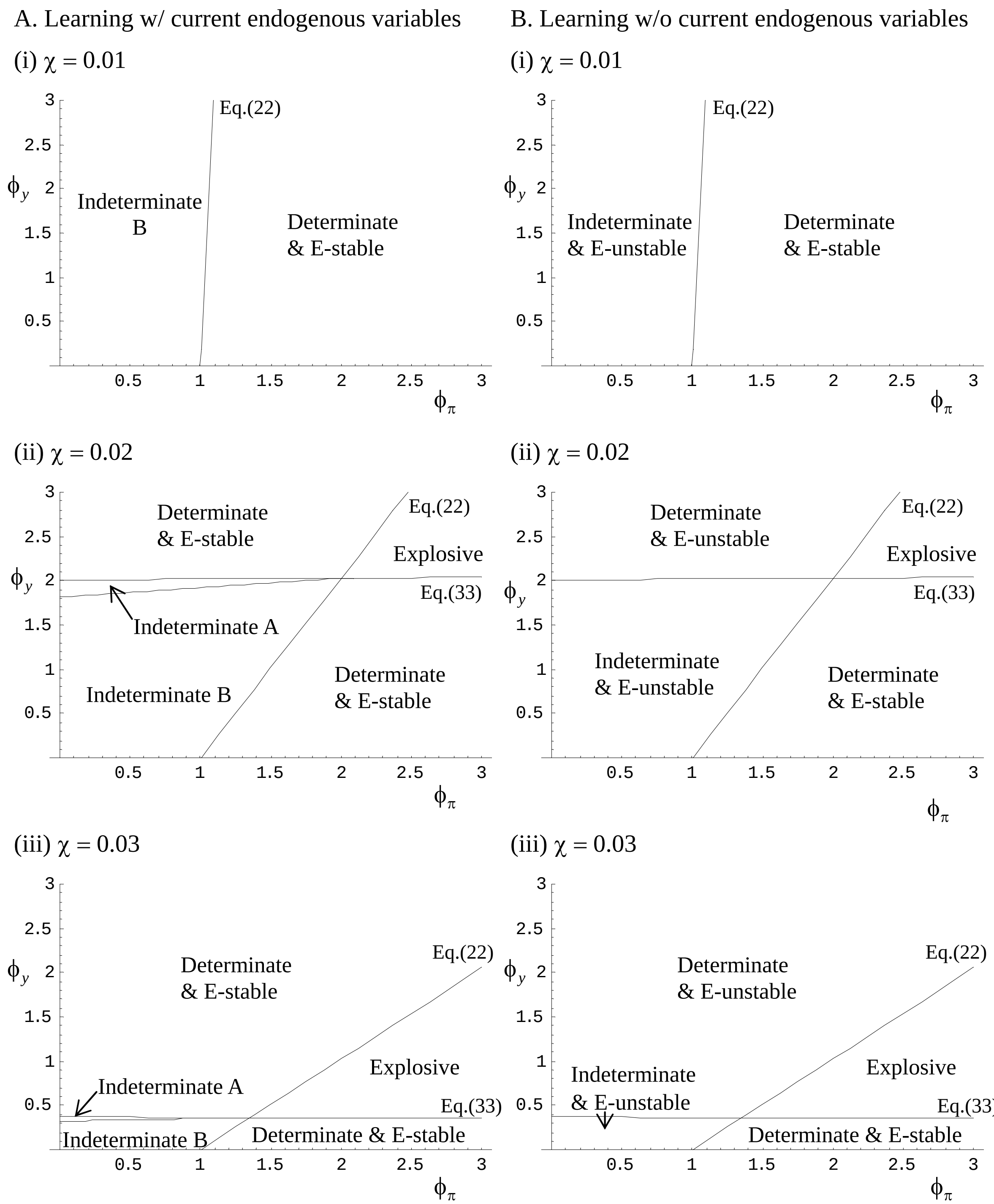

Figure 3. Regions of coefficients on Taylor rule ensuring determinacy of REE and E-stability of fundamental REE with Svensson-style CIA timing.

Note: In the region labeled as “Indeterminate A”, only the MOD fundamental REE is E-stable; while in the region labeled as “Indeterminate B”, only the non-MOD fundamental REE is E-stable.