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An Efficient Monte Carlo Method for a Large and Nongranular Credit Portfolio

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Abstract

It can be time consuming to evaluate the risk of a large credit portfolio with Monte Carlo simulation. This paper introduces a simple yet efficient Monte Carlo method where the portfolio is divided into subportfolios of obligors with large exposures and those with small exposures. Neglecting the idiosyncratic risks in the latter subportfolio, an approximation of value-at-risk for the entire portfolio is obtained in a short time. The new method is tested using sample portfolios of nongranular 5,000 exposures. The technique provides accurate credit value-at-risk with a computation time about one-fifteenth of ordinary Monte Carlo simulation. In addition to the improved computational efficiency, the method can also be used to specify the range of a subportfolio where idiosyncratic risks do not contribute to the value-at-risk of the entire portfolio. This may serve as important information when senior credit managers review the appropriateness and efficiency of internal risk management systems from the viewpoint of obligor's risk contribution.

* Any views expressed represent those of the author and not necessarily the Bank of Japan. Any remaining errors are the author's alone.

1. INTRODUCTION

Monte Carlo simulation is widely used to evaluate the risk of credit portfolios, however it is generally time consuming. Although the processing power of computers has grown year by year, it still takes anywhere from tens of minutes to several hours to calculate the credit value-at-risk of Japanese financial institutions. This is because the number of obligors in these credit portfolios can reach tens of thousands. This impedes the prompt reevaluation of risks in the entire portfolio when a credit manager observes or anticipates large changes in exposure, or restricts comparative analysis of the portfolio obtained by repeating Monte Carlo simulations with different parameter settings.[1]

One way to avoid this drawback of Monte Carlo simulation is to use an analytical solution for credit value-at-risk. It is well known that the Asymptotic Single Risk Factor model, which laid the foundation of risk weight functions for the internal rating-based approach in Basel II, is the analytical expression of value-at-risk using a Merton-type one-factor model assuming that the portfolio is infinitely fine grained.[2] As for the real-world portfolio with a finite number of obligors and a lumpy distribution of exposures, Gordy (2003) introduced the granularity adjustment technique and Martin and Wilde (2002) and Canabarro et al. (2003) derived a closed-form expression for value-at-risk in the Merton-type one-factor model. Pykhtin (2004) proposed an analytical approximation for value-at-risk in the multifactor Merton framework. However, these analytical approaches have a shortcoming in that they are not accurate when the portfolio is highly concentrated or when the default probabilities or correlations lie close to zero as shown in Tasche (2005), Ando (2005) and Higo (2006). Given the present circumstances, such analytical approximation does not provide a perfect alternative to Monte Carlo simulation methods.

This paper introduces a hybrid method which incorporates the strong points of both the Monte Carlo method and analytical approximation. The sketch of the new method is as follows. To start with, the portfolio is divided into two subportfolios, one consisting of larger exposures and the other with the remainder. Ordinary Monte Carlo procedures are then used to simulate the loss of the former. For the latter subportfolio, the idiosyncratic risks of the obligors are neglected and only expected losses conditioned on systemic factors are simulated. If the latter subportfolio is close to the infinitely fine-grained portfolio, the sum of the losses of the two subportfolios becomes a good proxy for the loss of the entire portfolio. The computational

[1] The computational complexity of Monte Carlo simulation does not matter when value-at-risk is calculated infrequently, e.g. monthly. However, if a bank aims for active credit portfolio management, where the adoption of new trades or dynamic hedging strategies is determined considering the effects on the value-at-risk of the entire portfolio, higher-speed technologies are needed.

[2] Basel Committee on Banking Supervision (2005).

complexity reduces as random variables representing idiosyncratic risk factors are not required for the latter subportfolio. The basic idea behind this method is very simple and implementation is relatively easy.

2. BASIC IDEA

2-1 COMPUTATIONAL COMPLEXITY OF A MONTE CARLO SIMULATION

This section briefly reviews the computational complexity of a Monte Carlo simulation. We consider a portfolio of loans to M obligors. Let us suppose that the losses on loans are driven by a vector of systematic factors $\vec{X} = \{X_1, X_2, \cdots, X_S\}$ and a vector of idiosyncratic factors $\vec{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_M\}$ for each obligor. In general, idiosyncratic factors $\vec{\varepsilon}$ are set to be independent with each other and also with $\vec{X}$.

Let us evaluate the value-at-risk of this portfolio with a Monte Carlo simulation where the number of paths is N . We generate random variables representing correlated systematic factors and independent idiosyncratic factors for each path, and then the computational complexity of the simulation is $O((S^2 + M) \times N)$.[3] Since the number of systematic factors S is fixed once one selects a risk model, the computational complexity of a simulation becomes $O(M \times N)$. This means that the time to finish the simulation doubles when the number of obligors M or the number of simulation paths N is doubled.

The easiest way to reduce the computational complexity of Monte Carlo simulation is to reduce the number of obligors M or the number of simulation paths N . It does not seem a good idea to reduce the number of paths because this will make the result of the simulation unstable. The starting point of the method used in this paper is to reduce the number of obligors M instead of N to ease the burden of Monte Carlo simulation.

2-2 EXPOSURE DISTRIBUTION IN REAL-WORLD CREDIT PORTFOLIO

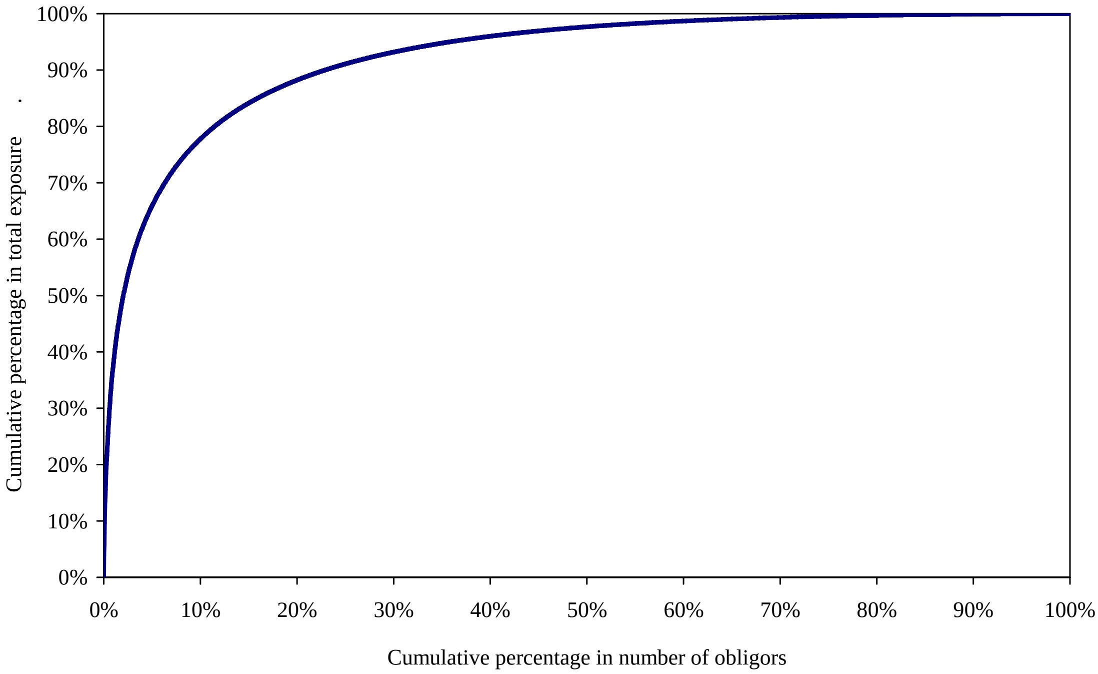
Chart 1 shows a sample cumulative distribution of exposures in Japanese Banks' credit portfolios.[4] The exposures are sorted in descending order of size. Chart 1 plots the cumulative

[3] To be precise, we have to consider the complexity of specifying the α percentile point, where α is confidence level, among N samples of portfolio loss. For simplicity, this computational complexity is omitted in the paper.

[4] The cumulative distribution is an average of ten arbitrarily chosen Japanese banks where the number of obligors ranges from about 3,000 to 30,000. The data are from a database in the Financial Systems and Bank Examination Department, Bank of Japan.

percentage of exposures in total exposure against the cumulative percentage of the number of obligors.

Chart 1. A sample cumulative distribution of credit exposures



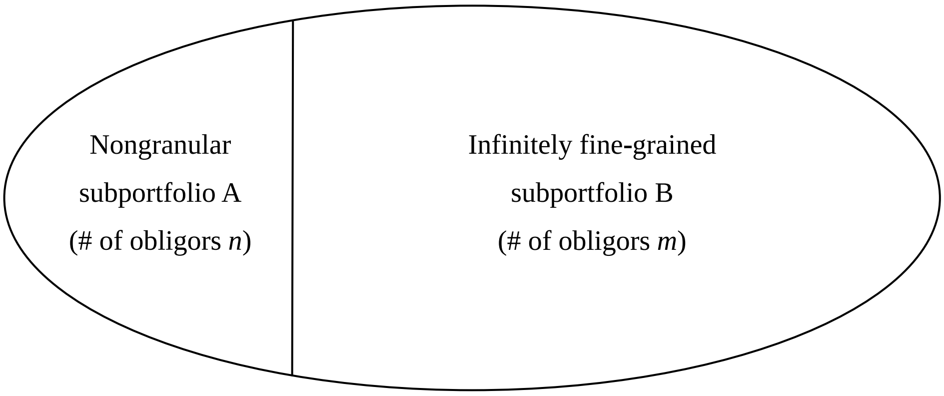
The chart shows that the largest 30% of obligors constitute more than 90% in total exposure and the remaining 70% of obligors make up less than 10% in total exposure. The exposure size of each obligor in the latter group is quite small compared with the total exposure. To a greater or lesser extent, heterogeneous exposure distributions are widely observed in many Japanese banks whose customers range from large companies to individuals.

It seems a natural idea that the major source of risk in the credit portfolio comes from large exposures and small exposures would not contribute very much to the portfolio risk given the heterogeneous exposure distribution shown in Chart 1.

2-3 A NONGRANULAR PORTFOLIO CONTAINING AN INFINITELY FINE-GRAINED SUBPORTFOLIO

Let us consider the value-at-risk of a hypothetical portfolio shown in Chart 2, containing an infinitely fine-grained subportfolio in which the size of each exposure is negligible.

Chart 2. A nongranular portfolio containing an infinitely fine-grained subportfolio



Note. Entire portfolio (subportfolio A plus B) is nongranular.

Let n denote the number of obligors in subportfolio A and m denote that in subportfolio B. The total number of obligors is $M = n + m$. The obligor i belongs to subportfolio A when $i = 1, 2, \cdots, n$ and belongs to subportfolio B when $i = n+1, n+2 \cdots, n+m = M$.

Let us suppose that Assumption 1 is satisfied for the exposures in nongranular subportfolio A.

$$\frac{EaD_i}{\sum_{i=1}^{M} EaD_i} = c_i > 0 \text{, where } c_i \text{ is constant.} \qquad \text{Assumption 1}$$

Note. EaD_i denotes gross exposure at default of obligor i .

Assumption 1 means that the n exposures in subportfolio A always take some positive and constant portion among the total exposure of the entire portfolio. Thus, we cannot neglect the sizes of exposures in subportfolio A.

On the other hand, subportfolio B is assumed to be infinitely fine grained and Assumptions 2 and 3 will be satisfied as $m \to \infty$.

$$\sum_{i=n+1}^{n+m} EaD_i \to \infty \, . \qquad \text{Assumption 2}$$

There exists $\zeta > 0$ such that $\dfrac{EaD_{n+m}}{\sum_{i=n+1}^{n+m} EaD_i} = O\left(m^{-\left(\frac{1}{2}+\zeta\right)}\right)$. Assumption 3

Assumptions 2 and 3 are to guarantee that the weight of the largest single exposure in subportfolio B vanishes to zero as the number of exposures increases.

Let L_i denote the gross loss from a loan to obligor i and L_M denote that of the entire portfolio. L_M is the sum of the loss of subportfolio A denoted by L_n^A, and the loss of subportfolio B denoted by L_m^B.

$$
\begin{aligned}
L_M &\equiv \sum_{i=1}^{M} L_i \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{n+m} L_i \\
&= L_n^A + L_m^B, \qquad (1)
\end{aligned}
$$

where $L_n^A \equiv \sum_{i=1}^{n} L_i$ and $L_m^B \equiv \sum_{i=n+1}^{n+m} L_i$.

Let us decompose L_m^B into a systematic component and an idiosyncratic component. The systematic component is defined as a conditional expectation of L_m^B on systematic factors $\vec{X}$ and denoted by $E\left[L_m^B \middle| \vec{X}\right]$. The idiosyncratic component is the remainder of L_m^B minus $E\left[L_m^B \middle| \vec{X}\right]$.

$$
L_M = L_n^A + \underbrace{E\left[L_m^B \middle| \vec{X}\right]}_{\text{Systematic component}} + \underbrace{L_m^B - E\left[L_m^B \middle| \vec{X}\right]}_{\text{Idiosyncratic component}} \qquad (2)
$$

Let $\vec{x}$ be the realization of $\vec{X}$. We then have Equation 3 given the assumption that the subportfolio B is infinitely fine grained.[5]

$$
L_m^B - E\left[L_m^B \middle| \vec{x}\right] \to 0 \quad \text{a.s. as} \quad m \to \infty . \qquad (3)
$$

Equation 3 means that the idiosyncratic component of L_m^B is diversified away as the number of obligors in subportfolio B increases and L_m^B will be dominated only by the systematic factors $\vec{X}$. With Equations 2 and 3 we have

[5] Gordy (2003).

$$L_m^B - E\left[L_m^B \middle| \vec{x}\right] = L_M - \left(L_n^A + E\left[L_m^B \middle| \vec{x}\right] \right) \tag{4}$$

$$L_M - \left(L_n^A + E\left[L_m^B \middle| \vec{x}\right] \right) \to 0 \text{ a.s. as } m \to \infty . \tag{5}$$

Let $q_\alpha(\bullet)$[6] denote α percentile of a random variable, then we have

$$q_\alpha(L_M) - q_\alpha\left(L_n^A + E\left[L_m^B \middle| \vec{X}\right] \right) \to 0 \text{ as } m \to \infty . \tag{6}$$

The proof is provided in the Appendix.[7] Equation 6 means that the α percentile of loss of this portfolio, i.e., value-at-risk at the confidence level α converges to that of sum of loss of subportfolio A and systematic component of subportfolio B as the number of obligors in B goes to infinity. In such a condition we will be able to neglect the idiosyncratic component of subportfolio B.

2-4 A MEASURE FOR 'SUFFICIENTLY' FINE-GRAINED SUBPORTFOLIO

Though the previous section assumes that the subportfolio B is infinitely fine grained, such a portfolio does not exist in the real world. In this section we consider a measure to specify a 'sufficiently' fine-grained subportfolio in a portfolio with a finite number of exposures, which serves as a proxy for an infinitely fine-grained subportfolio.

With regard to subportfolio B, Equation 7 will hold as $m \to \infty$ (at the same time $M \to \infty$), given Assumptions 2 and 3 in the previous section.[8]

$$\frac{\sum_{i=n+1}^{M} EaD_i^2}{\left(\sum_{i=n+1}^{M} EaD_i\right)^2} \to 0 . \tag{7}$$

Let us substitute the denominator of Equation 7 from the square of total exposure of subportfolio B to that of the entire portfolio. We then have

$$\frac{\sum_{i=n+1}^{M} EaD_i^2}{\left(\sum_{i=1}^{M} EaD_i\right)^2} \to 0 \text{ as } M \to \infty . \tag{8}$$

---

6 Refer to the Appendix for the definition of $q_\alpha(\bullet)$.

7 Strictly speaking, Equation 6 may not hold if the conditional expectation $E\left[L_m^B \middle| \vec{X}\right]$ is not a continuous function.

8 See Appendix A in Gordy (2003).

Therefore we may use the condition of Equation 9, the sum of exposure weights squared in subportfolio, to find a 'sufficiently' fine-grained subportfolio B.[9]

$$\frac{\sum_{i=n+1}^{M} EaD_i^2}{\left(\sum_{i=1}^{M} EaD_i\right)^2} = \sum_{i=n+1}^{M} w_i^2 \approx 0\,, \tag{9}$$

where $w_i = \dfrac{EaD_i}{\sum_{i=1}^{M} EaD_i}$ is an exposure weight of obligor i to the total exposure.

It is not yet known how small the sum of exposure weights squared in subportfolio B should be to have an accurate approximation for the value-at-risk of the entire portfolio. The following sections will test the new Monte Carlo method which divides a portfolio into subportfolio A and B and neglects the idiosyncratic risks in the latter (we hereafter call the new method the 'segmented Monte Carlo method').

The following two sections compare the value-at-risk calculated by the segmented Monte Carlo method with the ordinary Monte Carlo method based on the default-mode Merton-type credit risk models which are widely used by financial institutions. Section 3 considers the one-factor model and the Section 4 deals with the multifactor model.

3. NUMERICAL COMPARISONS ON A MERTON TYPE ONE-FACTOR MODEL

3-1 MODEL SPECIFICATION AND COMPUTATIONAL COMPLEXITY

In the Merton type one-factor model, the systematic factor $\vec{X}$ is not a vector but a single random variable X. The X and idiosyncratic factors for M obligors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_M$, follow independent standard normal distributions denoted by $N(0,1)$. We define Y_i, the corporate value of obligor i, as follows.

$$Y_i \equiv \sqrt{R_i}\,X + \sqrt{1-R_i}\,\varepsilon_i\,, \tag{10}$$

where $\sqrt{R_i}$ is a correlation coefficient between Y_i and X.

The default event occurs when the Y_i dips below a threshold level of $N^{-1}(PD_i)$, where

[9] Although Equation 7 is followed by Equation 9, the reverse is not always true. However, it can be said that the exposures in the subportfolio are quite small compared with total exposure when Equation 9 is satisfied. In practice, it would not cause a problem to specify subportfolio B using Equation 9.

$N^{-1}(\bullet)$ is the inverse of the cumulative distribution function of the standard normal distribution and PD_i denotes the default probability of obligor i. When obligor i defaults, it is supposed that we will have a gross loss of $EaD_i \times LGD_i$ (LGD_i denotes the loss ratio against exposure i at default).[10] L_i, the individual gross loss from exposure i, is defined as follows.

$$L_i \equiv EaD_i LGD_i 1_{\{Y_i < N^{-1}(PD_i)\}}. \tag{11}$$

Let L_M denote the gross loss from a portfolio of M obligors, L_M is the sum of individual L_i.

$$L_M \equiv \sum_{i=1}^{M} L_i . \tag{12}$$

The ordinary Monte Carlo method will generate N samples of L_M and picks up $q_\alpha(L_M)$, α percentile point of L_M, which will be the value-at-risk of this portfolio at confidence level α.

On the other hand, following the previous section we divide the portfolio into subportfolio A and B and let L'_M denote a new random variable which is the sum of gross loss from subportfolio A and the systematic component of loss from subportfolio B. In the one-factor Merton framework, we have

$$\begin{aligned}
L'_M &\equiv L_n^A + E\left[L_m^B \middle| X\right] \\
&= \sum_{i=1}^{n} L_i + E\left[\sum_{i=n+1}^{M} L_i \middle| X\right] \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} E\left[L_i \middle| X\right] \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} EaD_i LGD_i E\left[1_{\{Y_i \le N^{-1}(PD_i)\}} \middle| X\right] \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} EaD_i LGD_i \Pr\left\{Y_i < N^{-1}(PD_i) \middle| X\right\} \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} EaD_i LGD_i \Pr\left\{\sqrt{R_i} X + \sqrt{1-R_i}\,\varepsilon_i < N^{-1}(PD_i)\right\} \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} EaD_i LGD_i N\left(\frac{N^{-1}(PD_i) - \sqrt{R_i} X}{\sqrt{1-R_i}}\right).
\end{aligned} \tag{13}$$

[10] For simplicity, the LGD_i is assumed to a fixed, not a random, variable.

The segmented Monte Carlo method in the Merton type one-factor model generates the L'_M according to Equation 13 and picks up $q_\alpha(L'_M)$, i.e., the value-at-risk of L'_M instead of L_M.

Since $N\left(\frac{N^{-1}(PD_i)-\sqrt{R_i}X}{\sqrt{1-R_i}}\right)$ in Equation 13 is a function of PD_i and R_i, Equation 13 will be rewritten as follows when the obligors in subportfolio B are segmented into $K \times G$ homogeneous groups of PD_i and R_i.

$$L'_M = \sum_{i=1}^{n} L_i + \sum_{k=1}^{K}\sum_{g=1}^{G}\left(N\left(\frac{N^{-1}(PD(k))-\sqrt{R(g)}X}{\sqrt{1-R(g)}}\right)\sum_{i\in\{k,g\}}(EaD_i LGD_i)\right). \tag{14}$$

Note. $PD(k)$ and $R(g)$ denote the default probability of group k $(k=1,2,\cdots,K)$ and the correlation of group g $(g=1,2,\cdots G)$, respectively.

Equation 14 means that if $EaD_i \times LGD_i$, the gross loss from individual exposure, are summed for each homogeneous group, the computational complexity of the segmented Monte Carlo method with N paths will be $O(N \times (n + K \times G))$. This is $(n + K \times G)/M$ time of the ordinary Monte Carlo method of $O(N \times M)$. When the number of homogeneous groups $K \times G$ is sufficiently smaller than the number of total obligors M, $(n + K \times G)/M$ is approximately n/M and therefore the computational complexity of the segmented Monte Carlo method is proportional to the ratio of number of obligors in subportfolio A to the total number of obligors in the entire portfolio.

3-2 SAMPLE PORTFOLIO

The number of obligors in the sample portfolio is set at 5,000. The size of the 5,000 exposures are adjusted such that the cumulative distribution of exposures is the same as that in Chart 1, which shows the average distribution of real portfolios for the 10 Japanese Banks. The sum of exposures is set at 100 in order to make the numerical comparison easier.

In order to test the various types of portfolio, six hypothetical cases for the individual default probability PD_i, loss ratio at default LGD_i, and correlation R_i described in Chart 3 are used. Cases I, II, and III are portfolios which contains a large portion of obligors with low default probabilities (hereafter called ‘low-PD portfolios’.) On the other hand, Cases IV, V, and VI represent ‘high-PD portfolios’.[11] Each obligor’s loss ratio given default LGD_i is randomly set from 0.2 to 1.0. The correlation R_i is homogeneous among all obligors and takes values of

[11] Each obligor’s PD and LGD is randomly set according to the component percentages in parentheses in Chart 3.

0.01, 0.10, and 0.20.

Chart 3. Six hypothetical cases of sample portfolio

Case #	PD	LGD	R
I	0.03% (30%), 0.10% (25%), 0.50% (20%), 1.00% (15%), 5.00% (10%)	0.2 (20%), 0.4 (20%), 0.6 (20%), 0.8 (20%), 1.0 (20%)	0.01 (100%)
II			0.10 (100%)
III			0.20 (100%)
IV	0.03% (10%), 0.10% (15%), 0.50% (20%), 1.00% (25%), 5.00% (30%)		0.01 (100%)
V			0.10 (100%)
VI			0.20 (100%)

Note. Values in parentheses represent the percentage of the number of obligors.

3-3 SEGMENTATION OF PORTFOLIO AND COMPARISON OF VALUE-AT-RISK

Chart 4 shows how the value of Equation 9, i.e., the sum of exposure weights squared in the subportfolio B $\sum_{i=n+1}^{5000} w_i^2$, changes against the breakpoint index n $(n = 1,2,\cdots,5000)$ which divides portfolio into subportfolio A and B. The exposures are arranged in descending order where n is the index for the n th largest exposure, in other words, the smallest exposure in subportfolio A.

Chart 4. Sum of exposure weights squared in subportfolio B

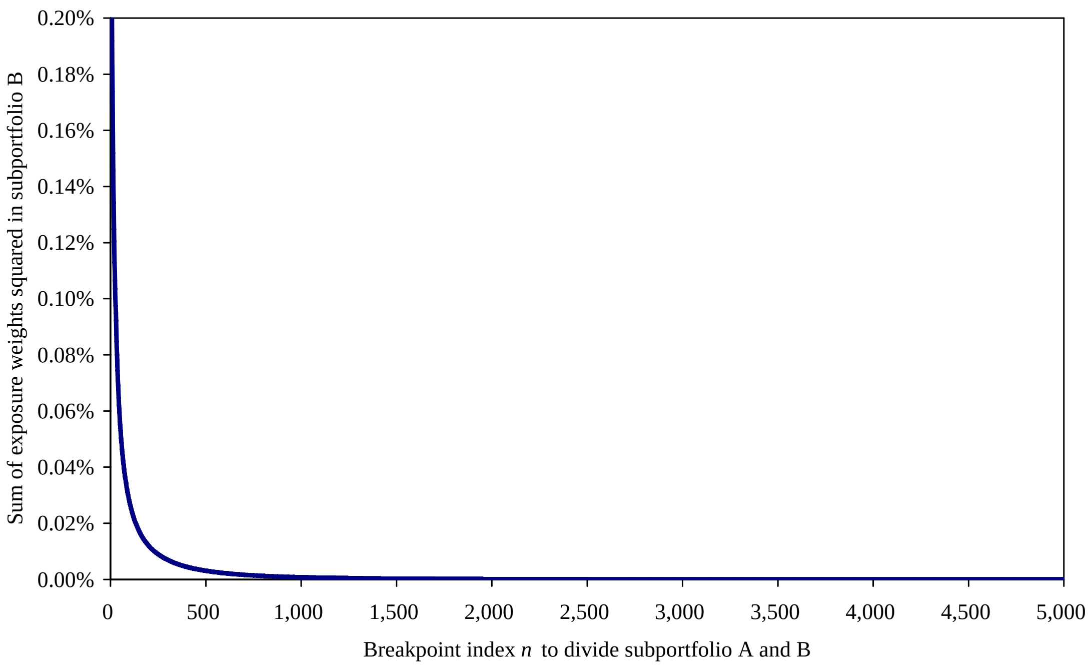


Chart 4 shows that the sum of exposure weights squared in subportfolio B quickly goes to zero as n increases. According to the value of sum of exposure weights squared in subportfolio B, we divide the sample portfolios into subportfolio A and B in six patterns as shown in Chart 5. The value-at-risk of the six patterns calculated using the segmented Monte Carlo method will be compared with those provided by the ordinary Monte Carlo method.

Chart 5. Six segmentation patterns of portfolio

Segment pattern	$\sum_{i=n+1}^{5000} w_i^2$	Number of exposures		Sum of exposures in subportfolio B
		Subportfolio A	Subportfolio B	
Ordinary M.C. method	——	5,000(100.0%)	0(0.0%)	0.0
Pattern 1	0.003%	513(10.3%)	4,487(89.7%)	21.8
Pattern 2	0.005%	375(7.5%)	4,625(92.5%)	27.0
Pattern 3	0.010%	231(4.6%)	4,769(95.4%)	35.4
Pattern 4	0.030%	92(1.8%)	4,908(98.2%)	51.6
Pattern 5	0.050%	56(1.1%)	4,944(98.9%)	60.1
Pattern 6	0.100%	26(0.5%)	4,974(99.5%)	72.2

Note 1. Values in the parentheses are component percentages.

Note 2. Total exposure of the sample portfolio is 100.0.

The number of paths for both ordinary and segmented Monte Carlo methods is set to one million to reduce numerical errors in simulation. The value-at-risk at the three confidence level of 95%, 99%, and 99.9% are calculated for each portfolio case. The value-at-risk of the ordinary Monte Carlo method is deemed to be ‘true’ and the accuracy of the segmented Monte Carlo method will be judged using the divergence from the true value-at-risk. In the author’s experience, the numerical error in the value-at-risk of a nongranular credit portfolio appears to be several percent and therefore the segmented Monte Carlo method is considered to be accurate if the divergence from the ordinary Monte Carlo method is within about 1%.[12]

3-4 RESULT

Chart 6 compares the value-at-risk of the segmented Monte Carlo method with the ordinary Monte Carlo method for each case of the sample portfolio. The meshed field means that absolute value of divergence is larger than one percent.

[12] The one percent criterion in this paper is based on the author’s experience and simulation system. Readers may choose a higher or lower criterion based on his/her experience and simulation system. To be statistically precise, it appears better to repeat the ordinary Monte Carlo simulation many times and specify the range of numerical error. However, it is very time consuming to do this with a million paths and therefore this paper depends on the subjective criteria from the author’s own experience.

Chart 6. Comparison of value-at-risk in the one-factor Merton model

Case I. Low-PD portfolio, LGD ranges from 0.20 to 1.00 and correlation is 0.01

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		1.241	——	1.666	——	2.216	——
Segmented M.C. method	Pattern 1	1.241	–0.02%	1.664	–0.14%	2.206	–0.43%
	Pattern 2	1.244	+0.25%	1.667	+0.09%	2.230	+0.64%
	Pattern 3	1.249	+0.59%	1.656	–0.62%	2.211	–0.23%
	Pattern 4	1.276	+2.80%	1.642	–1.46%	2.199	–0.77%
	Pattern 5	1.301	+4.82%	1.616	–2.98%	2.160	–2.51%
	Pattern 6	1.319	+6.22%	1.568	–5.90%	2.149	–2.98%

Case II. Low-PD portfolio, LGD ranges from 0.20 to 1.00 and correlation is 0.10

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		1.194	——	1.933	——	3.065	——
Segmented M.C. method	Pattern 1	1.197	+0.24%	1.935	+0.06%	3.080	+0.50%
	Pattern 2	1.192	–0.17%	1.938	+0.26%	3.090	+0.80%
	Pattern 3	1.194	–0.06%	1.935	+0.08%	3.101	+1.19%
	Pattern 4	1.179	–1.27%	1.913	–1.07%	3.048	–0.54%
	Pattern 5	1.166	–2.34%	1.888	–2.34%	3.014	–1.67%
	Pattern 6	1.107	–7.28%	1.822	–5.77%	2.915	–4.89%

Case III. Low-PD portfolio, LGD ranges from 0.20 to 1.00 and correlation is 0.20

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		1.442	——	2.659	——	4.896	——
Segmented M.C. method	Pattern 1	1.445	+0.19%	2.671	+0.43%	4.881	–0.31%
	Pattern 2	1.446	+0.27%	2.687	+1.06%	4.919	+0.45%
	Pattern 3	1.438	–0.31%	2.675	+0.58%	4.893	–0.06%
	Pattern 4	1.429	–0.93%	2.645	–0.54%	4.902	+0.12%
	Pattern 5	1.421	–1.47%	2.626	–1.25%	4.806	–1.84%
	Pattern 6	1.403	–2.75%	2.608	–1.92%	4.758	–2.83%

Case IV. High-PD portfolio, LGD ranges from 0.20 to 1.00 and correlation is 0.01

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		2.512	——	3.320	——	4.293	——
Segmented M.C. method	Pattern 1	2.519	+0.25%	3.323	+0.11%	4.287	–0.16%
	Pattern 2	2.517	+0.18%	3.319	–0.03%	4.301	+0.18%
	Pattern 3	2.509	–0.12%	3.313	–0.19%	4.282	–0.26%
	Pattern 4	2.498	–0.59%	3.298	–0.65%	4.262	–0.72%
	Pattern 5	2.480	–1.29%	3.281	–1.18%	4.253	–0.93%
	Pattern 6	2.430	–3.28%	3.194	–3.79%	4.105	–4.40%

Case V. High-PD portfolio, LGD ranges from 0.20 to 1.00 and correlation is 0.10

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		3.377	——	5.103	——	7.667	——
Segmented M.C. method	Pattern 1	3.371	–0.18%	5.088	–0.31%	7.623	–0.57%
	Pattern 2	3.374	–0.08%	5.113	+0.19%	7.652	–0.20%
	Pattern 3	3.355	–0.65%	5.092	–0.22%	7.632	–0.45%
	Pattern 4	3.349	–0.82%	5.064	–0.78%	7.582	–1.10%
	Pattern 5	3.349	–0.83%	5.050	–1.05%	7.433	–3.05%
	Pattern 6	3.310	–1.98%	4.989	–2.24%	7.392	–3.58%

Case VI. High-PD portfolio, LGD ranges from 0.20 to 1.00 and correlation is 0.20

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		3.872	——	6.659	——	11.087	——
Segmented M.C. method	Pattern 1	3.874	+0.04%	6.692	+0.49%	11.107	+0.17%
	Pattern 2	3.869	–0.08%	6.636	–0.34%	11.068	–0.17%
	Pattern 3	3.873	+0.01%	6.645	–0.21%	11.090	+0.03%
	Pattern 4	3.861	–0.29%	6.652	–0.11%	11.104	+0.15%
	Pattern 5	3.846	–0.68%	6.583	–1.14%	10.993	–0.85%
	Pattern 6	3.822	–1.30%	6.566	–1.40%	11.033	–0.49%

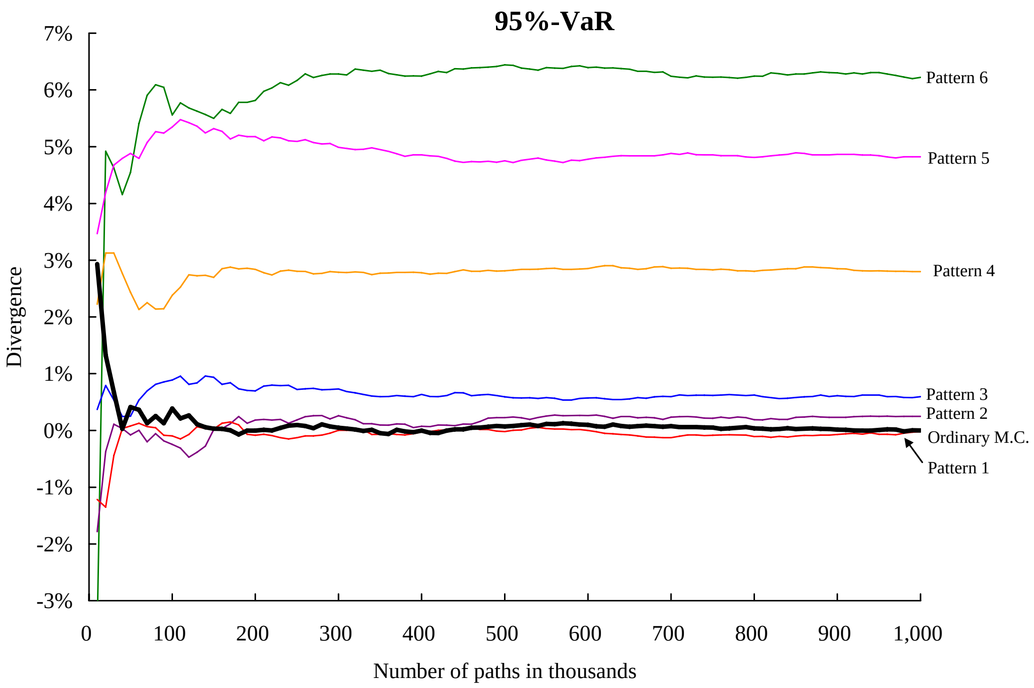
Examining Cases I to VI, the divergence becomes larger as the number of pattern increases, i.e., as the number of exposures in the subportfolio B increases. Based on the experiential criteria of one percent divergence, it seems that the segmented Monte Carlo method of Pattern 3 gives an accurate approximation for the value-at-risk obtained using the ordinary Monte Carlo method. In segmentation Pattern 3, the number of obligors is 4,769, which is 95.4% of all obligors, and the sum of exposure weights squared in subportfolio B is 0.01%.

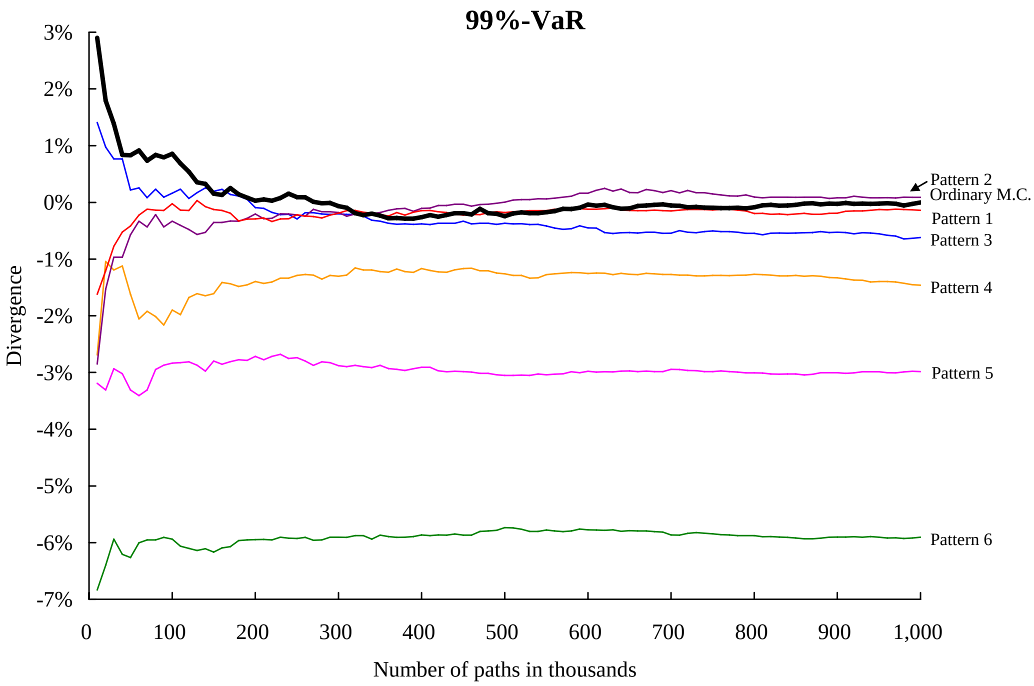
Chart 7 shows how the value-at-risk in Case I evolves. This suggests that the value-at-risk using the segmented Monte Carlo method of Patterns 1, 2, and 3 becomes close to the 'true' value-at-risk using the ordinary Monte Carlo method as the number of paths increases.

Chart 8 compares the cumulative distributions of portfolio loss by the segmented Monte Carlo method of Pattern 3 and 4 with the ordinary Monte Carlo method in Case I. It is observed that the distribution by the segmented Monte Carlo method of Pattern 4 apparently differs from that by the ordinary Monte Carlo simulation. On the other hand, the distribution by the segmented Monte Carlo method of Pattern 3 is almost the same as that by the ordinary Monte Carlo method except in the lower left corner of the chart. This suggests that the segmented Monte Carlo method of Pattern 3 accurately approximates not only the value-at-risk at the confidence levels of 95%, 99%, and 99.9%, but also the whole distribution of portfolio loss.

Chart 7. Convergence of value-at-risk in Case I

Note. The divergence in the chart is the rate of divergence against the value-at-risk by the ordinary Monte Carlo method when the 1,000,000th path is finished.





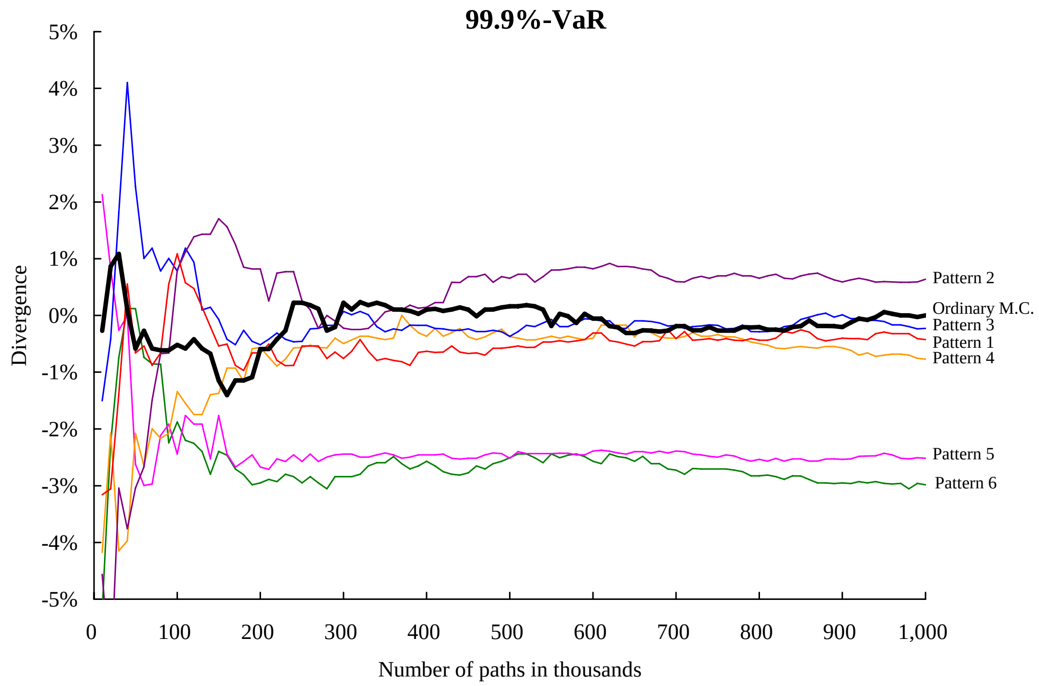


Chart 8. Cumulative distribution of portfolio loss in Case I

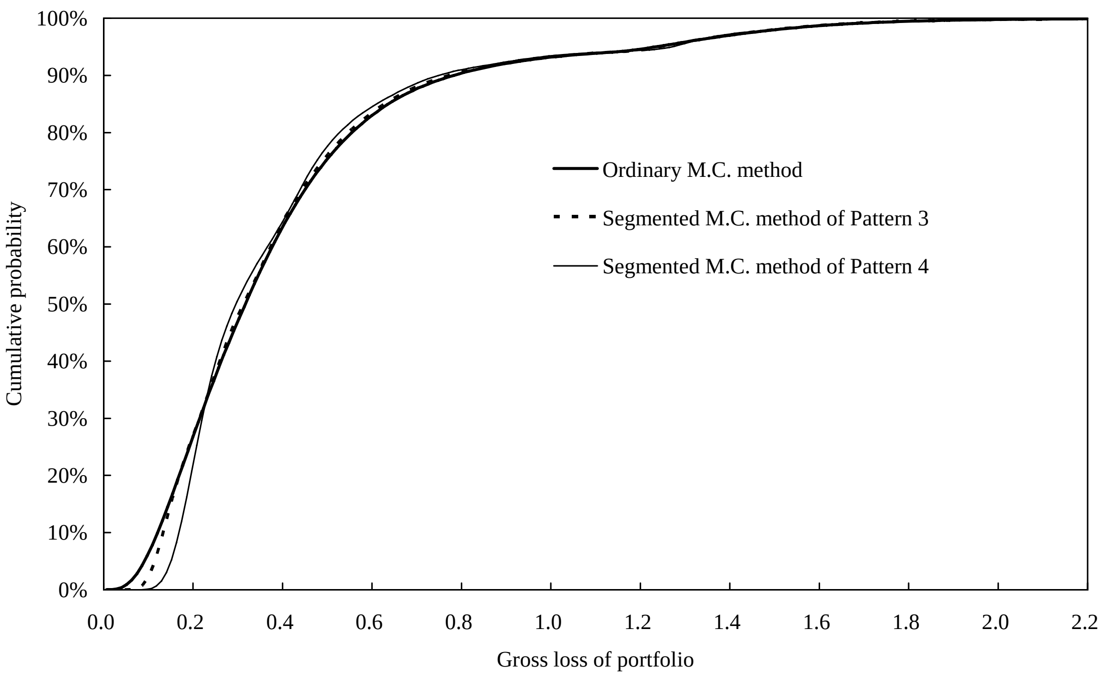


Chart 9 shows the computation time for the segmented Monte Carlo method. With respect to segmentation Pattern 3, the number of exposures in subportfolio A is 231 and takes up only 4.6% of the total number of exposures, and the computation time is about one-fifteenth (6.3%)

of the ordinary Monte Carlo method.[13]

Chart 9. Computation time of the segmented Monte Carlo method (average of Cases I to IV)

Segment pattern	$\sum_{i=n+1}^{5000} w_i^2$	Number of exposures		Computation time
		Subportfolio A	Subportfolio B	
Ordinary M.C. method	——	5,000(100.0%)	0(0.0%)	1.000
Pattern 1	0.003%	513(10.3%)	4,487(89.7%)	0.119
Pattern 2	0.005%	375(7.5%)	4,625(92.5%)	0.092
Pattern 3	0.010%	231(4.6%)	4,769(95.4%)	0.063
Pattern 4	0.030%	92(1.8%)	4,908(98.2%)	0.036
Pattern 5	0.050%	56(1.1%)	4,944(98.9%)	0.030
Pattern 6	0.100%	26(0.5%)	4,974(99.5%)	0.024

Note 1. Values in the parentheses are component percentages in the number of obligors.

Note 2. Computation time is standardized against the ordinary Monte Carlo method.

3-5 COMPARISON WITH AN ANALYTICAL APPROXIMATION METHOD

Chart 10 compares the value-at-risk obtained by the granularity adjustment method proposed by Martin and Wilde (2002) and Canabarro et al. (2003) with those using the ordinary Monte Carlo method of one million paths.

Chart 10. Value-at-risk by the granularity adjustment method

Case #	95%-VaR			99%-VaR			99.9%-VaR		
	Ordinary M.C. method	Granularity adj. method		Ordinary M.C. method	Granularity adj. method		Ordinary M.C. method	Granularity adj. method	
			Divergence			Divergence			Divergence
I	1.241	1.509	+21.6%	1.666	2.010	+20.6%	2.216	2.600	+17.4%
II	1.194	1.203	+0.7%	1.933	1.890	–2.2%	3.065	2.986	–2.6%
III	1.442	1.432	–0.8%	2.659	2.644	–0.6%	4.896	4.859	–0.8%
IV	2.512	2.891	+15.1%	3.320	3.695	+11.3%	4.293	4.654	+8.4%
V	3.377	3.349	–0.8%	5.103	5.008	–1.9%	7.667	7.458	–2.7%
VI	3.872	3.874	+0.0%	6.659	6.628	–0.5%	11.087	11.043	–0.4%

It takes only a few seconds to calculate value-at-risk by the granularity adjustment method and the divergence from the ordinary Monte Carlo simulation is less than 1% for Cases III and VI, where the correlation is higher than in the other cases. However, as the correlation becomes

[13] With the sample portfolio of 5,000 exposures with one million paths, the gross computation time of an ordinary Monte Carlo method implemented by the author is 112 minutes. It takes only seven minutes when the segmented Monte Carlo of Pattern 3 is applied. The simulation program is written using C++ language and runs on a personal computer with Windows 2000, Celeron 2.00GHz CPU, and 521MB RAM.

lower, the accuracy worsens. The divergence is about 20% in Case I where the portfolio contains a large portion of obligors whose probabilities of default are low and the correlation is 0.01.

The segmented Monte Carlo method is inferior to the granularity adjustment method from the viewpoint of computational time. However, an advantage of the segmented Monte Carlo method is that it gives an accurate approximation of value-at-risk, even if a portfolio contains a large portion of low-PD obligors or where the correlation is close to zero. In these instances, the granularity adjustment method does not work well.

4. NUMERICAL COMPARISONS ON A MERTON-TYPE MULTIFACTOR MODEL

4-1 MODEL SPECIFICATION AND COMPUTATIONAL COMPLEXITY

In this section we test the segmented Monte Carlo method on the default-mode Merton-type multifactor model. The systematic factors $\vec{X}$ are assumed to be S random variables which follow the multidimensional standard normal distribution.

$$\vec{X} = \{X_1, X_2, \cdots, X_S\} \sim N(0, Q). \tag{15}$$

Note. The Q represents correlation matrix between systematic factors.

$$Q = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,S} \\ \rho_{S,1} & 1 & & \vdots \\ \vdots & & \ddots & \rho_{S-1,S} \\ \rho_{S,1} & \cdots & \rho_{S,S-1} & 1 \end{pmatrix}, \text{ where } \rho_{i,j} = \rho_{j,i}.$$

$s = 1, 2, \cdots, S$ represents the index for the sector that each obligor belongs to, e.g., industries, countries, or region. We define Y_i, the corporate value of obligor i belonging to the sector s, by the systematic factor $X_{s\{i \in s\}}$ and an idiosyncratic factor ε_i which is independent of the $X_{s\{i \in s\}}$.

$$Y_i \equiv \sqrt{R_{i,s}}\, X_s + \sqrt{1 - R_{i,s}}\, \varepsilon_i, \tag{16}$$

where $\sqrt{R_{i,s}}$ is an intrasector correlation coefficient between Y_i and X_s.

As is in the previous section for the one-factor model, idiosyncratic factors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_M$

follow independent standard normal distributions and gross loss of exposure i, $EaD_i \times LGD_i$[14], will arise if Y_i is below the threshold of $N^{-1}(PD_i)$. The definitions of L_i and L_M are the same with those in the previous section.

$$L_i \equiv EaD_i LGD_i 1_{\{Y_i < N^{-1}(PD_i)\}}. \tag{17}$$

$$L_M \equiv \sum_{i=1}^{M} L_i \,. \tag{18}$$

Let L'_M denote a random variable which is the sum of gross loss from subportfolio A and the systematic component of loss from subportfolio B. In the multifactor Merton framework, we have

$$\begin{aligned}
L'_M &\equiv L_n^A + E\left[L_m^B \middle| \vec{X}\right] \\
&= \sum_{i=1}^{n} L_i + E\left[\sum_{i=n+1}^{M} L_i \middle| \vec{X}\right] \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} E\left[L_i \middle| \vec{X}\right] \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} EaD_i LGD_i E\left[1_{\{Y_i \le N^{-1}(PD_i)\}} \middle| \vec{X}\right] \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} EaD_i LGD_i \Pr\left\{Y_i < N^{-1}(PD_i) \middle| X_{s\{i \in s\}}\right\} \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} EaD_i LGD_i \Pr\left\{\sqrt{R_{i,s}} X_s + \sqrt{1-R_{i,s}}\,\varepsilon_i < N^{-1}(PD_i)\right\} \\
&= \sum_{i=1}^{n} L_i + \sum_{i=n+1}^{M} EaD_i LGD_i N\left(\frac{N^{-1}(PD_i) - \sqrt{R_{i,s}} X_s}{\sqrt{1-R_{i,s}}}\right).
\end{aligned} \tag{19}$$

The segmented Monte Carlo method in the multifactor Merton framework generates L'_M according to Equation 19 and find the α percentile of L'_M, which is the proxy for the value-at-risk of the entire portfolio. Since $N\left(\frac{N^{-1}(PD_i) - \sqrt{R_{i,s}} X_s}{\sqrt{1-R_{i,s}}}\right)$ depends on s, PD_i and $R_{i,s}$, we can rewrite Equation 19 as follows when the obligors in subportfolio B are

[14] As is in the previous section, the LGD is assumed to be a fixed value and not a random variable.

segmented into $S \times K \times G$ homogeneous groups with respect to s, PD_i and $R_{i,s}$.

$$L'_M = \sum_{i=1}^{n} L_i + \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{g=1}^{G}\left(N\left(\frac{N^{-1}(PD(k)) - \sqrt{R_s(g)}X_s}{\sqrt{1-R_s(g)}} \right) \sum_{i\in\{s,k,g\}}(EaD_i LGD_i) \right). \tag{20}$$

Note 1. S, K and G are the number of sectors, values of PD_i, and $R_{i,s}$, respectively.

Note 2. $PD(k)$ denotes the default probability of group k.
Note 3. $R_s(g)$ denotes correlation of group g onto X_s.

Equation 20 means that if the $EaD_i \times LGD_i$ are summed for each homogeneous group, the computational complexity of the segmented Monte Carlo method with N paths will be $O(N \times (n + S \times K \times G))$. Compared with the one-factor model, the number of homogeneous groups becomes S times that in Equation 14 and this would reduce the computational efficiency. However, if the total number of groups, $S \times K \times G$, is sufficiently smaller than the number of total obligors, M, the computational complexity of segmented Monte Carlo method is proportional to the ratio of number of obligors in subportfolio A to the total number of obligors as shown in the previous section.

4-2 SAMPLE PORTFOLIO

The number of exposures in the sample portfolio and the distribution of exposures are the same as those in the previous section. The number of sectors representing industries is set at 10. The correlation matrix for the systematic factors $\vec{X}$ is provided in Chart 11, which was generated using 10 monthly TOPIX industrial indices from 2001 to 2005.

Chart 11. Correlation matrix of systematic factors

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	1	0.38	0.44	0.48	0.59	0.23	0.50	0.38	0.45	0.44
X_2	0.38	1	0.48	0.40	0.61	0.42	0.57	0.67	0.48	0.63
X_3	0.44	0.48	1	0.59	0.70	0.30	0.54	0.48	0.56	0.60
X_4	0.48	0.40	0.59	1	0.64	0.26	0.51	0.41	0.48	0.45
X_5	0.59	0.61	0.70	0.64	1	0.53	0.83	0.43	0.73	0.77
X_6	0.23	0.42	0.30	0.26	0.53	1	0.55	0.32	0.41	0.50
X_7	0.50	0.57	0.54	0.51	0.83	0.55	1	0.45	0.85	0.85
X_8	0.38	0.67	0.48	0.41	0.43	0.32	0.45	1	0.29	0.50
X_9	0.45	0.48	0.56	0.48	0.73	0.41	0.85	0.29	1	0.80
X_{10}	0.44	0.63	0.60	0.45	0.77	0.50	0.85	0.50	0.80	1

With regard to the portfolio conditions, six hypothetical cases for the individual default probability PD, loss ratio given default LGD, sector s, and intrasector correlation R_s are used. Following the previous section, Cases I, II, and III represent low-PD portfolios and Cases IV, V, and VI represent high-PD portfolios.

Chart 12. Six hypothetical cases of sample portfolio

Case #	PD	LGD	Sector	R_s
I	0.03% (30%), 0.10% (25%), 0.50% (20%), 1.00% (15%), 5.00% (10%)	0.2 (20%), 0.4 (20%), 0.6 (20%), 0.8 (20%), 1.0 (20%)	10% for each sector	0.01 (100%)
II				0.10 (100%)
III				0.20 (100%)
IV	0.03% (10%), 0.10% (15%), 0.50% (20%), 1.00% (25%), 5.00% (30%)			0.01 (100%)
V				0.10 (100%)
VI				0.20 (100%)

Note. Values in parentheses represent the percentage of the number of obligors.

4-3 SEGMENTATION PATTERN AND COMPARISON OF VALUE-AT-RISK

The segmentation pattern is the same as those in Chart 5 for the one-factor model. The procedure to compare the segmented Monte Carlo method with the ordinary Monte Carlo method follows that in the previous section.

4-4 RESULT

Chart 13 compares the value-at-risk of the segmented Monte Carlo with the ordinary Monte Carlo for each case of sample portfolio. As in Chart 6, the meshed field means that absolute value of divergence is larger than one percent.

Chart 13. Comparison of value-at-risk in multifactor Merton model

Case I. Low-PD portfolio, LGD ranges from 0.20 to 1.00 and intrasector correlation is 0.01

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		1.333	——	1.816	——	2.427	——
Segmented M.C. method	Pattern 1	1.334	+0.09%	1.811	–0.23%	2.420	–0.27%
	Pattern 2	1.338	+0.44%	1.819	+0.21%	2.431	+0.18%
	Pattern 3	1.338	+0.37%	1.812	–0.22%	2.421	–0.25%
	Pattern 4	1.358	+1.89%	1.818	+0.11%	2.402	–1.03%
	Pattern 5	1.384	+3.87%	1.835	+1.07%	2.387	–1.62%
	Pattern 6	1.410	+5.83%	1.636	–9.90%	2.333	–3.85%

Case II. Low-PD portfolio, LGD ranges from 0.20 to 1.00 and intrasector correlation is 0.10

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		1.899	——	2.807	——	4.011	——
Segmented M.C. method	Pattern 1	1.903	+0.23%	2.804	–0.12%	3.998	–0.34%
	Pattern 2	1.899	+0.00%	2.806	–0.04%	4.036	+0.61%
	Pattern 3	1.897	–0.10%	2.811	+0.14%	4.020	+0.22%
	Pattern 4	1.893	–0.34%	2.801	–0.20%	4.000	–0.28%
	Pattern 5	1.884	–0.81%	2.788	–0.67%	3.994	–0.42%
	Pattern 6	1.848	–2.68%	2.747	–2.15%	3.966	–1.13%

Case III. Low-PD portfolio, LGD ranges from 0.20 to 1.00 and intrasector correlation is 0.20

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		1.442	——	2.354	——	3.840	——
Segmented M.C. method	Pattern 1	1.442	–0.03%	2.354	+0.00%	3.841	+0.02%
	Pattern 2	1.444	+0.14%	2.350	–0.17%	3.826	–0.34%
	Pattern 3	1.437	–0.37%	2.345	–0.38%	3.834	–0.15%
	Pattern 4	1.430	–0.82%	2.332	–0.94%	3.833	–0.18%
	Pattern 5	1.428	–0.93%	2.328	–1.12%	3.841	+0.03%
	Pattern 6	1.390	–3.59%	2.262	–3.90%	3.710	–3.38%

Case IV. High-PD portfolio, LGD ranges from 0.20 to 1.00 and intracorrelation is 0.01

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		2.443	——	3.276	——	4.357	——
Segmented M.C. method	Pattern 1	2.437	–0.26%	3.271	–0.17%	4.342	–0.35%
	Pattern 2	2.436	–0.30%	3.263	–0.42%	4.337	–0.44%
	Pattern 3	2.432	–0.44%	3.266	–0.31%	4.340	–0.39%
	Pattern 4	2.422	–0.87%	3.241	–1.07%	4.303	–1.23%
	Pattern 5	2.412	–1.28%	3.232	–1.35%	4.305	–1.18%
	Pattern 6	2.429	–0.59%	3.199	–2.37%	4.286	–1.63%

Case V. High-PD portfolio, LGD ranges from 0.20 to 1.00 and intrasector correlation is 0.10

		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary Monte Carlo		2.621	——	4.084	——	5.967	——
Segmented Monte-Carlo	Pattern 1	2.620	–0.06%	4.086	+0.06%	5.948	–0.31%
	Pattern 2	2.619	–0.08%	4.100	+0.40%	5.923	–0.72%
	Pattern 3	2.618	–0.13%	4.087	+0.09%	5.941	–0.42%
	Pattern 4	2.585	–1.40%	4.054	–0.73%	5.909	–0.96%
	Pattern 5	2.579	–1.63%	4.052	–0.78%	5.836	–2.19%
	Pattern 6	2.537	–3.22%	4.013	–1.73%	5.843	–2.08%

Case VI. High-PD portfolio, LGD ranges from 0.20 to 1.00 and intrasector correlation is 0.20

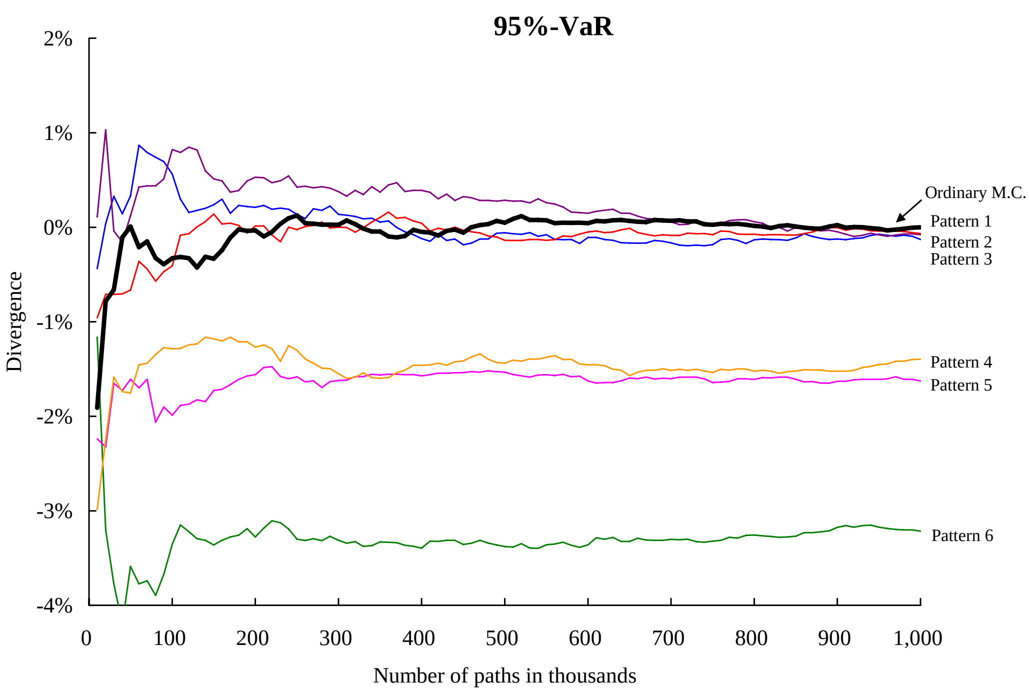
		95%-VaR		99%-VaR		99.9%-VaR	
			Divergence		Divergence		Divergence
Ordinary M.C. method		3.397	——	5.548	——	8.473	——
Segmented M.C. method	Pattern 1	3.402	+0.14%	5.559	+0.20%	8.565	+1.08%
	Pattern 2	3.395	–0.06%	5.549	+0.01%	8.549	+0.90%
	Pattern 3	3.391	–0.20%	5.523	–0.46%	8.480	+0.07%
	Pattern 4	3.375	–0.66%	5.531	–0.32%	8.475	+0.02%
	Pattern 5	3.363	–1.02%	5.529	–0.34%	8.444	–0.34%
	Pattern 6	3.316	–2.41%	5.450	–1.77%	8.361	–1.32%

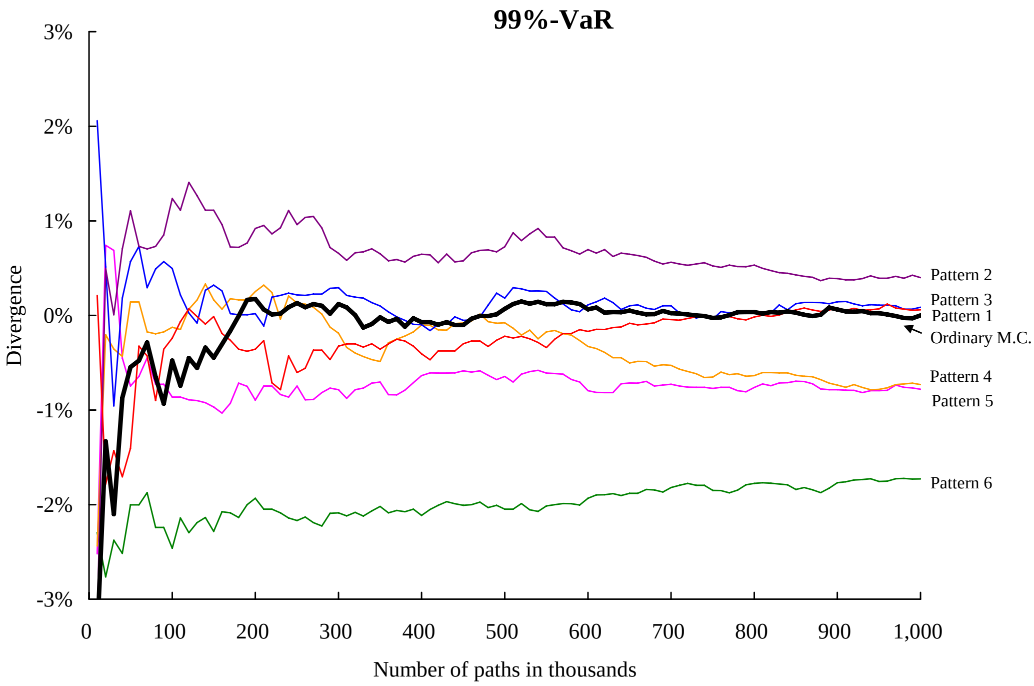
The results are similar to those in the previous section. Looking through Cases I to VI, the segmented Monte Carlo method of Pattern 3 gives an accurate approximation for the value-at-risk using the ordinary Monte Carlo method.

Chart 14 shows how the value-at-risk in Case V evolves. This suggests that the value-at-risk by the segmented Monte Carlo method of Patterns 1, 2 and 3 get close to the true value-at-risk obtained by the ordinary Monte Carlo method as the number of paths increases.

Chart 14. Convergence of value-at-risk in Case V

Note. The ‘divergence’ in the chart means the rate of divergence against the value-at-risk by the ordinary Monte Carlo method when the 1,000,000th path is finished.





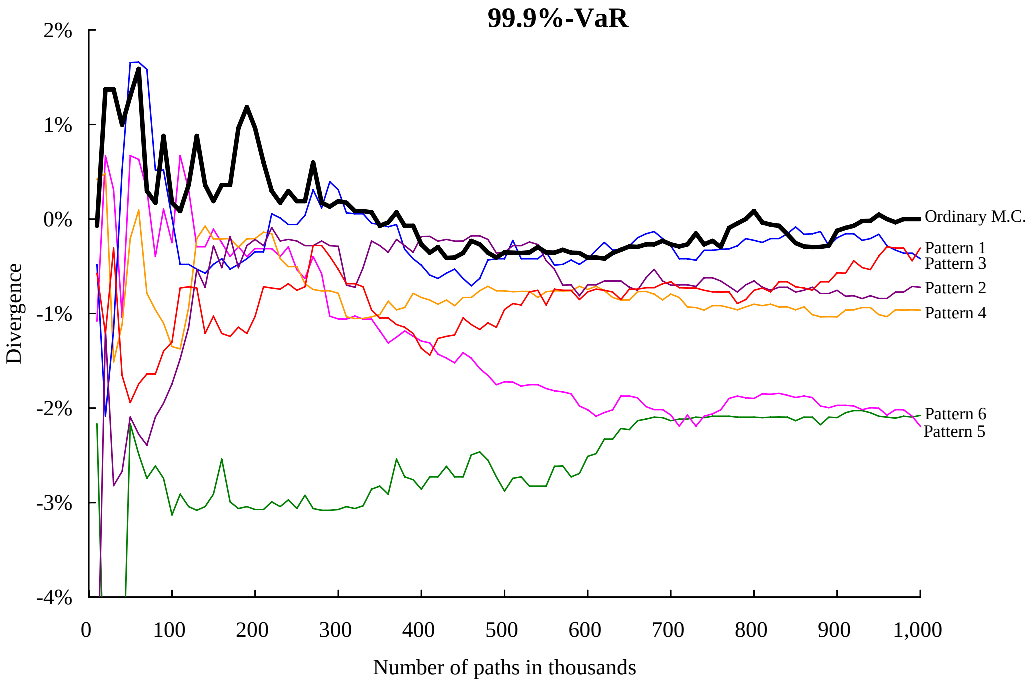
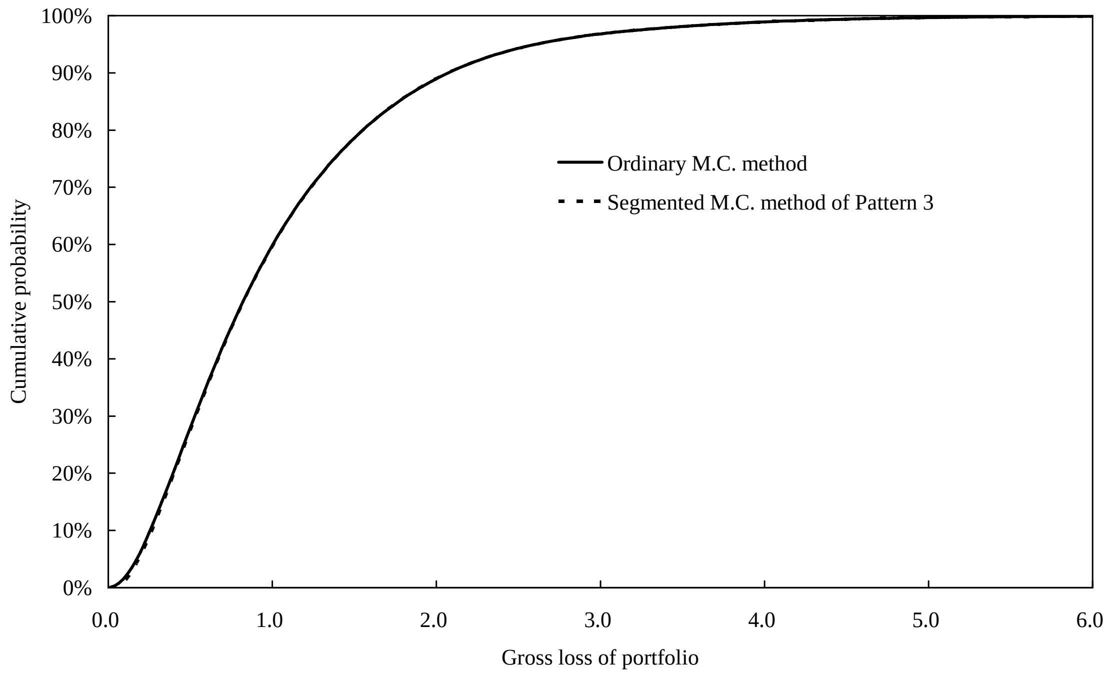


Chart 15 compares the cumulative distribution of portfolio loss in the segmented Monte Carlo method of Pattern 3 with the ordinary Monte Carlo method in Case V. It seems difficult to distinguish the two curves. This suggests that the segmented Monte Carlo method of Pattern 3 accurately approximates not only the value-at-risk at the confidence levels of 95%, 99%, and 99.9%, but also the whole distribution of portfolio loss.

Chart 15. Cumulative distribution of portfolio loss in Case V



The computation time of the segmented Monte Carlo method of Pattern 3 is about one-fifteenth (6.9%) of the ordinary Monte Carlo method.

Chart 16. Computation time of the segmented Monte Carlo method (average of Cases I to V)

Segment pattern	$\sum_{i=n+1}^{5000} w_i^2$	Number of exposures		Computation time
		Subportfolio A	Subportfolio B	
Ordinary M.C. method	——	5,000 (100.0%)	0 (0.0%)	1.000
Pattern 1	0.003%	513 (10.3%)	4,487 (89.7%)	0.124
Pattern 2	0.005%	375 (7.5%)	4,625 (92.5%)	0.097
Pattern 3	0.010%	231 (4.6%)	4,769 (95.4%)	0.069
Pattern 4	0.030%	92 (1.8%)	4,908 (98.2%)	0.042
Pattern 5	0.050%	56 (1.1%)	4,944 (98.9%)	0.035
Pattern 6	0.100%	26 (0.5%)	4,974 (99.5%)	0.029

Note 1. Values in the parentheses are the component percentages in the number of obligors.

Note 2. Computation time is standardized on the ordinary Monte Carlo method.

4-5 COMPARISON WITH AN ANALYTICAL APPROXIMATION METHOD

Chart 17 compares the value-at-risk obtained by the multifactor adjustment method proposed by Pykhtin (2004) with those of the ordinary Monte Carlo method with one million paths.

Chart 17. Value-at-risk by the multifactor adjustment method

Case #	95%-VaR			99%-VaR			99.9%-VaR		
	Ordinary M.C. method	Multifactor adj. method		Ordinary M.C. method	Multifactor adj. method		Ordinary M.C. method	Multifactor adj. method	
			Divergence			Divergence			Divergence
I	1.333	2.033	+52.55%	1.816	2.728	+50.24%	2.427	3.525	+45.27%
II	1.899	1.976	+4.04%	2.807	2.808	+0.03%	4.011	3.934	–1.93%
III	1.442	1.437	–0.35%	2.354	2.315	–1.65%	3.840	3.725	–2.99%
IV	2.443	3.174	+29.93%	3.276	4.098	+25.07%	4.357	5.168	+18.63%
V	2.621	2.693	+2.73%	4.084	3.814	–6.60%	5.967	5.404	–9.43%
VI	3.397	3.414	+0.49%	5.548	5.293	–4.60%	8.473	8.113	–4.25%

The multifactor adjustment method gives a good approximation for the value-at-risk by the ordinary Monte Carlo method in Cases III and VI, where the intrasector correlation is relatively high. However, it becomes quite inaccurate when the intrasector correlation becomes close to zero in Cases I and IV. It can be said that an advantage of the segmented Monte Carlo method is that it gives an accurate approximation of value-at-risk even if the multifactor adjustment method does not work well.

From the viewpoint of computation complexity, the multifactor adjustment method proposed by Pykhtin (2004) has one disadvantage in that it explicitly evaluates the correlation of all pairs of idiosyncratic factors and thus the computation complexity becomes $O\left(M^2\right)$, where M is the number of obligors.[15] Therefore the computational efficiency of the multifactor adjustment method rapidly decreases as M increases. The segmented Monte Carlo method could be less time consuming than the multifactor adjustment method if the number of obligors is large.[16]

5. CONCLUSION

5-1 PROS AND CONS OF THE SEGMENTED MONTE CARLO METHOD

The basic idea of the segmented Monte Carlo method is quite simple. It is naturally expected that the idiosyncratic factors of small exposures would have little to do with the risk of the

[15] In the original multifactor model, it is assumed that the idiosyncratic factors are independent as explained in Section 4-1. However, the multifactor adjustment method proposed by Pykhtin (2004) transforms the original multifactor model into a “comparable one-factor model” and the independency of the idiosyncratic factors will not hold.

[16] With the program implemented by the author, it takes about nine minutes to calculate the value-at-risk of a sample portfolio with the multifactor adjustment method. Since we have to reevaluate value-at-risk for different confidence levels, it takes about 30 minutes to provide the value-at-risk at the three different confidence levels. On the other hand, it takes seven minutes to apply the segmented Monte Carlo method of Pattern 3 to the sample portfolio, where the number of paths is one million. The segmented Monte Carlo method can simultaneously evaluate the value-at-risk at three different confidence levels as is in the ordinary Monte Carlo method.

entire portfolio. To apply the method, a portfolio is divided into two parts: subportfolio A constituting larger exposures and subportfolio B comprising smaller exposures. In the segmented Monte Carlo method, the ordinary procedure will be applied to the former but only the expected value of loss given systematic factors will be simulated in regard to the latter.

The new method is tested on sample portfolios which reflect the granularity of Japanese banks and shows a nice approximation for the value-at-risk obtained with the ordinary Monte Carlo method, while the computation time is about one-fifteenth of the ordinary Monte Carlo method. Compared with the analytical solutions proposed by Canabarro et al. (2003) and Pykhtin (2004), the segmented Monte Carlo method has an advantage in that it is accurate, even in conditions where these analytical solutions do not work well.[17] In addition, the segmented Monte Carlo method is easy to implement if the ordinary Monte Carlo method has been based on the Merton framework. As discussed in Section 3 and 4, the ordinary Monte Carlo method will be transformed to the segmented Monte Carlo method by changing program codes to sum up losses given default of small exposures by the homogeneous groups of sector, default probability and correlation.[18]

Although this paper shows the numerical examples only in the Merton framework, the basic idea in Section 2 is applicable to any credit risk model that uses systematic and idiosyncratic factors. If we can specify the expectation of loss from subportfolio B conditioned on systematic factors, $E\left[L_m^B \middle| \vec{X}\right]$, it is anticipated that the segmented Monte Carlo method will contribute to the computational efficiency to a greater or lesser extent.[19]

On the other hand, the segmented Monte Carlo method may have a weak point in that we do not know the relation between the sum of exposure weights squared in subportfolio B, $\sum_{i=n+1}^{M} w_i^2$, and the accuracy of value-at-risk. What we know is only that the smaller

[17] The segmented Monte Carlo method appears easier than the analytical solutions in handling the problem of accuracy in approximation. The accuracy of analytical solutions tested in this paper becomes worse when a portfolio has a large portion of obligors whose default probabilities and correlations on systematic factors are low. When we observe inaccurate results from these analytical solutions, there is no good remedy to help improve accuracy since we can not change the default probabilities or correlation. On the other hand, if we observe inaccurate results from the segmented Monte Carlo method, the accuracy can be improved by narrowing the range of subportfolio B at the cost of computational efficiency.

[18] However, the computational efficiency may decrease if the sector, default probability and correlation are segmented finely and the number of homogeneous groups of obligors in subportfolio B becomes large.

[19] Strictly speaking, the segmented Monte Carlo method might lead to inaccurate approximations if the $E\left[L_m^B \middle| \vec{X}\right]$ is not a continuous function.

$\sum_{i=n+1}^{M} w_i^2$ is better. The numerical examples in this paper show that the segmented Monte Carlo method of Pattern 3, where $\sum_{i=n+1}^{M} w_i^2$ is 0.01%, gives accurate approximations, however, an appropriate value of $\sum_{i=n+1}^{M} w_i^2$ may depend on conditions in the portfolios, the credit risk models used or the allowances for the accuracy of value-at-risk. Numerical comparisons as in Section 3 and 4 will be needed before we apply the segmented Monte Carlo method to a new portfolio. An appropriate value of $\sum_{i=n+1}^{M} w_i^2$ will be determined via such analyses, considering the trade-off relationship between computational efficiency and the degree of accuracy. Furthermore, the appropriateness of the value of $\sum_{i=n+1}^{M} w_i^2$ should be tested periodically since any portfolio will change as time goes by.

5-2 SECONDARY EFFECT: IDENTIFYING THE RETAIL POOL FROM THE VIEWPOINT OF PORTFOLIO RISK

One weak point of the segmented Monte Carlo method is that we need to experientially specify the 'sufficiently fine-grained' subportfolio B based on numerical analyses of risk. However, this process may at the same time give us important information which relates to the appropriateness of risk management systems inside banks.

The subportfolio B in this paper appears to correspond to the retail pool. The retail pool in general is defined by some criteria on the types of obligors and loan, e.g., individual, corporate size, mortgage or consumer loan, and sometimes by the dollar amount of the loan. It can be said that determining the range of subportfolio B in the process of segmented Monte Carlo simulation is to identify the retail pool from the viewpoint of risk, i.e., the risk contribution of idiosyncratic factors to the risk of the entire credit portfolio.

The analyses in this paper identified that subportfolio B comprises the smaller 95% of obligors, whose total exposure constitute about 30% of the entire portfolio exposure. This means that fluctuations in the expected loss for this subportfolio by systematic factors are more important than the idiosyncratic risk of each obligor when we consider the risk of the entire portfolio. However, a risk manager may excessively examine the idiosyncratic factors of these small obligors and place less emphasis in the analysis on the evolution of average loss.[20] It may be useful to check the appropriateness and efficiency of the risk management system based on analysis of the range of subportfolio B. At the same time, one may wish to check if obligors belonging to subportfolio A, who have a nonnegligible impact on the risk of entire portfolio, are segmented into the retail pool by mistake.

The segmented Monte Carlo method proposed in this paper has a secondary merit in that one can review the appropriateness and efficiency of the internal credit management system from the viewpoint of the contribution of idiosyncratic factors to the risk of the entire portfolio, in addition to improving the computational efficiency of Monte Carlo simulation.

[20] To avoid any misunderstanding, the author does not say that we can fully neglect the idiosyncratic factors of small obligors. It is essential for retail business to properly judge the credit worthiness of every obligor. What the author means is that analysis of the average loss of subportfolio B becomes more important for managers who are in charge of the risk control for the entire portfolio.

APPENDIX. PROOF OF EQUATION 6[21]

Let $\vec{\varepsilon}^{(A)} = \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ and $\vec{\varepsilon}^{(B)} = \{\varepsilon_{n+1}, \varepsilon_{n+2}, \cdots, \varepsilon_M\}$ denote vectors of idiosyncratic factors for the obligors in subportfolio A and B, respectively. The gross loss of subportfolio A, L_n^A, and that of entire portfolio, L_M, are the functions of systematic factors and the above idiosyncratic factors.

$$L_n^A = L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right)$$

$$L_M = L_M\left(\vec{X}, \vec{\varepsilon}^{(A)}, \vec{\varepsilon}^{(B)}\right)$$

Let $\vec{x}$ and $\vec{e}^{(A)}$ be the realization of $\vec{X}$ and $\vec{\varepsilon}^{(A)}$, respectively. From Equation 5, Equation A-1 below is satisfied as $m \to \infty$ (at the same time $M \to \infty$ since $M = m + n$).

$$L_M\left(\vec{x}, \vec{e}^{(A)}, \vec{\varepsilon}^{(B)}\right) - \left(L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] \right) \to 0 \quad \text{a.s.} \tag{A-1}$$

Almost sure convergence implies convergence in probability, so for all $\vec{x}$, $\vec{e}^{(A)}$ and $\zeta > 0$,

$$\Pr\left(\left| L_M\left(\vec{x}, \vec{e}^{(A)}, \vec{\varepsilon}^{(B)}\right) - \left(L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] \right) \right| < \zeta \,\middle|\, \vec{x}, \vec{e}^{(A)} \right) \to 1 \text{ as } m \to \infty . \tag{A-2}$$

If F_M is the cumulative distribution function of L_M, then Equation A-2 implies

$$\begin{aligned} &F_M\left(L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] + \zeta \mid \vec{x}, \vec{e}^{(A)} \right) \\ &- F_M\left(L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] - \zeta \mid \vec{x}, \vec{e}^{(A)} \right) \to 1 \end{aligned} \tag{A-3}$$

as $m \to \infty$. Because F_M is bounded in $[0,1]$, we must have the following two equations.

$$F_M\left(L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] + \zeta \mid \vec{x}, \vec{e}^{(A)} \right) \to 1 \text{ as } m \to \infty . \tag{A-4}$$

$$F_M\left(L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] - \zeta \mid \vec{x}, \vec{e}^{(A)} \right) \to 0 \text{ as } m \to \infty . \tag{A-5}$$

Let S_m^+ denote the set of $\vec{x}$ and $\vec{e}^{(A)}$ such that $L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right]$ is less than or equal

[21] The proof in this appendix is based on Appendix B in Gordy (2003).

to its α percentile,[22] i.e.,

$$S_m^+ \equiv \left\{ \vec{x}, \vec{e}^{(A)} \middle| L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] \le q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) \right\}.$$

By construction,

$$\Pr\left(\vec{x}, \vec{e}^{(A)} \in S_m^+\right) \ge \alpha . \tag{A-6}$$

By the usual rules for conditional probability, we have

$$\begin{aligned} & F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) + \zeta \right) \\ &= F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) + \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \in S_m^+ \right) \cdot \Pr\left(\vec{X}, \vec{\varepsilon}^{(A)} \in S_m^+\right) \\ &\quad + F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) + \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \notin S_m^+ \right) \cdot \Pr\left(\vec{X}, \vec{\varepsilon}^{(A)} \notin S_m^+\right) \\ &\ge F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) + \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \in S_m^+ \right) \cdot \Pr\left(\vec{X}, \vec{\varepsilon}^{(A)} \in S_m^+\right) \\ &\ge F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) + \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \in S_m^+ \right) \alpha . \end{aligned} \tag{A-7}$$

For all $\vec{x}, \vec{e}^{(A)} \in S_m^+$, we have

$$\begin{aligned} 1 &\ge F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) + \zeta \middle| \vec{x}, \vec{e}^{(A)} \right) \\ &\ge F_M\left(L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] + \zeta \middle| \vec{x}, \vec{e}^{(A)} \right) \to 1 \end{aligned} \tag{A-8}$$

as $m \to \infty$, so the dominated convergence theorem implies that

$$F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) + \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \in S_m^+ \right) \to 1 \quad \text{as} \quad m \to \infty . \tag{A-9}$$

Plugging Equation A-9 into A-7, we have

$$F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) + \zeta \right) \ge \alpha \quad \text{as} \quad m \to \infty . \tag{A-10}$$

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[22] For an arbitrary random variable Z, α percentile is defined as $q_\alpha(Z) \equiv \inf\left\{ z \middle| \Pr(Z \le z) \ge \alpha \right\}$.

The other half of the proof follows similarly. Define S_m^- as

$$S_m^- \equiv \left\{ \vec{x}, \vec{e}^{(A)} \middle| L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] \geq q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) \right\}$$

so that

$$\Pr\left(\vec{x}, \vec{e}^{(A)} \in S_m^-\right) \geq 1 - \alpha \,. \tag{A-11}$$

Then

$$\begin{aligned} &F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) - \zeta \right) \\ &= F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) - \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \in S_m^- \right) \cdot \Pr\left(\vec{X}, \vec{\varepsilon}^{(A)} \in S_m^-\right) \\ &\quad + F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) - \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \notin S_m^- \right) \cdot \Pr\left(\vec{X}, \vec{\varepsilon}^{(A)} \notin S_m^-\right) \\ &\leq \alpha + F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) - \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \in S_m^- \right) \cdot \Pr\left(\vec{X}, \vec{\varepsilon}^{(A)} \in S_m^-\right). \end{aligned} \tag{A-12}$$

For all $\vec{x}, \vec{e}^{(A)} \in S_m^-$, we have

$$\begin{aligned} 0 &\leq F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) - \zeta \middle| \vec{x}, \vec{e}^{(A)} \right) \\ &\leq F_M\left(L_n^A\left(\vec{x}, \vec{e}^{(A)}\right) + E\left[L_m^B \middle| \vec{x}\right] - \zeta \middle| \vec{x}, \vec{e}^{(A)} \right) \to 0 \end{aligned} \tag{A-13}$$

as $m \to \infty$. So the dominated convergence theorem implies that

$$F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) - \zeta \middle| \vec{X}, \vec{\varepsilon}^{(A)} \in S_m^- \right) \to 0 \quad \text{as} \quad m \to \infty \,. \tag{A-14}$$

Plugging Equation A-14 into A-12, we have

$$F_M\left(q_\alpha\left(L_n^A\left(\vec{X}, \vec{\varepsilon}^{(A)}\right) + E\left[L_m^B \middle| \vec{X}\right] \right) - \zeta \right) \leq \alpha \quad \text{as} \quad m \to \infty \,. \tag{A-15}$$

With the assumption that the $E\left[L_m^B \middle| \vec{X}\right]$ is continuous, Equation A-10 and A-15 ensures that Equation 6 will hold.

$$q_\alpha\left(L_M\right) - q_\alpha\left(L_n^A + E\left[L_m^B \middle| \vec{X}\right] \right) \to 0 \quad \text{as} \quad m \to \infty \,.$$

REFERENCES

Basel Committee on Banking Supervision, "An Explanatory Note on the Basel II IRB Risk Weight Functions," July, 2005 (available at http://www.bis.org)

Canabarro, E., E. Picoult, and T. Wilde, "Analysing counterparty risk," *Risk*, 16(9), 2003, pp. 117–122.

Gordy, M., "A risk-factor model foundation for rating based bank capital rules," *Journal of Financial Intermediation*, 12(3), July 2003, pp.199–232

Martin, R. and T. Wilde, "Unsystematic credit risk," *Risk*, 15(11), 2002, pp.123–128

Pykhtin, M., "Multi-factor adjustment," *Risk*, 17(3), 2004, pp.85–90

Tasche, D., "Calculating credit risk capital charges with the one-factor model," *Journal of Risk*, 7, 2005, pp.85–101

Ando, Y., "Analytical methods for credit risk evaluation: limiting loss distribution and granularity adjustment," *Kinyukenkyu* 24, additional vol. 1, Bank of Japan, 2005, pp.39–120 (in Japanese)

Higo, H., "Analytical approximation method for UL contribution and its application for credit risk management," presentation material for seminars hosted by Center for Advanced Financial Technology, Bank of Japan, 2006 (in Japanese and available at http://www.boj.or.jp/type/release/zuiji_new/fsc0608a.htm)