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EQUILIBRIUM INTEREST RATE AND THE YIELD CURVE IN A LOW INTEREST RATE ENVIRONMENT [#]

Hibiki Ichiue[♦] and Yoichi Ueno[*]

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Abstract

Equilibrium nominal interest rates are useful indicators for both monetary policy authorities and market players. However, there are few studies which estimate Japan's equilibrium rate because of its persistent low interest rate. We overcome this challenge by using survey forecasts of interest rates and macroeconomic variables to estimate a two-factor yield curve model, which takes the bound of zero interest into account. We found that: 1) the equilibrium rate is roughly approximated with the long-run expected nominal output growth rate; 2) the Bank of Japan's commitments successfully lowered yields even at zero interest; and 3) the term premium of 10-year yield has had a downtrend since 2004.

JEL classification: E43, E44, E52, G12

Keywords: Equilibrium interest rate, Black's model of interest rates as options, Monetary policy, Macro-finance, Survey data

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1. Introduction

Equilibrium nominal interest rates possibly take on different meanings depending on the context. However, when the monetary policy authorities and bond market participants communicate with one another, equilibrium rates typically mean long-run target levels of short-term interest rates. For instance, the FOMC statements released by the Federal Reserve from June 2004 to November 2005 had continued to say, “With underlying inflation expected to be contained, the Committee believes that policy accommodation can be removed at a pace that is likely to be measured.” The statement was interpreted by market players as meaning that the Committee believed that the federal funds target rate was below the equilibrium rate. The Bank of Japan (BOJ)’s monetary policy—after the termination of quantitative monetary easing policy (QMEP) in March 2006—has also been interpreted as normalization of its target rate to an equilibrium interest rate.

Equilibrium interest rates are so important for monetary policy authorities, not only because the equilibrium rate is the level of interest rate that the authorities should adjust to in the long-run, but also because economic theory suggests the equilibrium rate reflect information on the long-run expected output growth and inflation. The market players also have major concerns, since the equilibrium rate influences the yield curve through the expected path of monetary policy. The problem is that the equilibrium rate is unobservable and should be estimated.

Roughly speaking, two types of approaches have been applied to estimate equilibrium interest rates in the literature. The first employs macroeconomic theories, models, and variables, and this approach has an advantage: it may help to understand the structural relationship between equilibrium rates and macroeconomic variables. Nonetheless, this measure of equilibrium rate also has a limitation: the measure depends heavily on the economic assumptions. The other approach employs the yield curve data. Since the implied forward rates reflect expected levels of future short-term interest rates, this approach enables us to estimate the market perception of equilibrium rates in a more direct manner, although unobservable risk premia should be removed using some yield curve models.

This paper estimates Japan's equilibrium rate based on the second approach. We use the yield curve data, because the yield curve reflects a monetary policy stance even when the short-term interest rate—the most important monetary policy instrument and conventionally used for the empirical analyses of monetary policy in the other countries—is bounded at zero, as was the case in Japan. We also take an advantage of the first approach to examine how the equilibrium rate is related to long-run expected economic growth and inflation. In this exercise, we use survey forecasts of average real GDP growth and CPI inflation between approximately the next 5 to 10 years as the proxies for the long-run expectations.

Many papers in the literature explicitly or implicitly study the equilibrium interest rate in the United States using the yield curve data, although they have different purposes and use different terms, such as central tendency or endpoint, for what corresponds to the equilibrium interest rate. Balduzzi, Das, and Foresi (1998) examine a term structure model with two factors including a central tendency, a time-varying mean for the short rate to revert. Bomfin (2003) interprets the central tendency as the long-run monetary policy expectation. Kozicki and Tinsley (2001) argue that the downtrend of long-term bond yields in the 1980s and 1990s was caused by a shift in the equilibrium rate, which reflected a decline in the implicit inflation target perceived by market participants. Hördahl, Tristani, and Vestin (2006), Rudebusch and Wu (2004), and Bekaert, Cho, and Moreno (2005) add a perceived inflation target as a latent factor in their no-arbitrage structural macro-finance models. Gürkaynack, Sack, and Swanson (2005) find that a macroeconomic model with a perceived inflation target can explain their empirical findings, the impulse response functions of long-term forward rates. In sum, these studies suggest that the perceived inflation target influence the yield curve through the equilibrium interest rate in the United States.

On the other hand, just a few studies estimate the equilibrium interest rate using the yield curve in Japan.[1] One important reason for this is the difficulty of empirical analyses due to the persistent low interest rate environment and the BOJ's untraditional policies.[2] Although the BOJ terminated the QMEP and began to raise its target on the overnight call rate in 2006, the target is still very low. In addition, we need to use the time-series data even before the termination of the QMEP to secure sufficient samples for reliable empirical exercises. The problem is that typical models fail to explain the unusual convex shape of the yield curve, which is flatter for shorter maturities; this reflects the expectations that the BOJ will not raise its target rate for a considerable

[1] Oda and Suzuki (2007) are exceptional, and estimate the equilibrium rate using a structural macro-finance model, which is restricted with a new Keynesian model. On the other hand, our approach is less restricted, and enables the model to fit the yield data well.

[2] The BOJ adopted the zero interest rate policy (ZIRP), under which the call rate was lowered to the lowest possible, from February 1999 to August 2000. In April 1999, the BOJ also announced a commitment that the ZIRP would be maintained until the deflationary concerns subsided. In March 2001, the BOJ started its QMEP with another commitment that the QMEP would be maintained at least until the year-on-year percent change of CPI stayed sustainably positive.

period.[3]

To overcome this limitation, we apply Black's (1995) model of interest rates as options. In the model, the spot nominal rate equals a shadow rate if the shadow rate is positive, and zero otherwise. Thus the nominal rate can be interpreted as an option of the shadow rate, and the model takes into account the zero bound of interest rates and the nonlinear relationship between the spot rate and the yield curve. While Black only suggests the idea of the model, Gorovoi and Linetsky (2004) construct the model in detail. Ueno, Baba, and Sakurai (2006) calibrate a version of Gorovoi and Linetsky's model by fitting to the yield curve data at each point in time independently with their five-year sample since the start of the QMEP. They confirm the usefulness of the model in explaining the convex shape of yield curve under the zero interest rate, although their result shows that the parameter corresponding to the equilibrium interest rate has widely fluctuated, which is in opposition to the model's assumption of stable parameters.

Our earlier paper, Ichiue and Ueno (2006), uses time-series data to estimate the time-invariant deep parameters of a generalized one-factor Black's model. The one-factor model successfully replicates the convex shape of short to middle-term term structure of interest rates even with time-invariant parameters, but the fit of the model to the longer-term yields is less accurate. These studies imply that the equilibrium interest rate has fluctuated in Japan, and is another important factor that potentially drives long-term yields. This observation motivates us to adopt a Black's model with two factors: the shadow rate and the equilibrium interest rate. This generalization contributes to a better

[3] For instance, Bernanke, Reinhart, and Sack's (2005) macro-finance model fails to explain the convex shape of the yield curve in Japan, and thus their empirical results provide limited insights on monetary policy.

fit, and enables us to extract the market perception of equilibrium rate and to examine its relationship with the yield curve.

Another problem in studying Japan's data is a possible structural break in the mid-1990s. For instance, Miyao (2000) provides empirical supports for a structural break in the Japanese economy in 1995. Even in the United States, Rudebusch and Wu (2004) argue that the behavior of the yield curve has a structural break in the mid-1980s, when the volatilities of macroeconomic variables sharply dropped or the so-called great moderation began. Kim and Orphanides (2005) recommend using a shorter sample to avoid the structural break, and propose a method to estimate no-arbitrage term structure models using survey data of short-term interest rates as an additional source of information, which contributes to overcoming the small-sample problem. We follow Kim and Orphanides in using survey data to avoid the possible break in Japan.

The rest of this paper is organized as follows. Section 2 reviews our two-factor model of interest rates as options. Section 3 describes the estimation method using the survey data. Section 4 evaluates the model's fit and discusses the estimated parameters. Section 5 investigates the time-series properties of the estimated equilibrium interest rate and the shadow rate, and then examines how these factors influence the yield curve. Section 6 concludes the paper.

2. A Black's Model of Interest Rates as Options with a Central Tendency

This section reviews our two-factor model, which is a generalization of a one-factor Black's model proposed by Gorovoi and Linetsky (2004). The nominal spot rate i_t, which is defined as an instantaneous interest rate per annum, has a zero bound constraint and should be nonnegative. The spot rate is specified as a function of a shadow rate x_t:

$$i_t = \max(x_t, 0). \tag{1}$$

That is, the nominal spot rate is equal to the shadow rate if the shadow rate is positive, and zero otherwise.

The two factors in the model, an equilibrium interest rate θ_t and the shadow rate, are assumed to follow stochastic processes:

$$d\theta_t = \kappa_\theta(\theta - \theta_t)dt + \sigma_\theta dW_{\theta,t} + \sigma_{\theta x} dW_{x,t}, \tag{2}$$

$$dx_t = \kappa_x(\theta_t - x_t)dt + \sigma_x dW_{x,t}, \tag{3}$$

where θ is the time-invariant steady state level of the shadow rate or spot rate. κ_θ and κ_x are the rate of mean reversion of the equilibrium rate toward its steady state level, and that of the shadow rate toward the equilibrium rate, respectively. $W_{\theta,t}$ and $W_{x,t}$ are orthogonal standard Brownian motions. The former is called the equilibrium rate shock, and its innovation can be interpreted as an unanticipated change in long-run expected output growth or inflation. On the other hand, the latter is called the shadow rate shock, and its innovation can be interpreted as an unanticipated change in monetary policy, which may reflect an endogenous response to a surprising shift in current inflation or just an exogenous change in monetary policy stance. σ_θ and σ_x are the volatility parameters of the innovations. The other parameter $\sigma_{\theta x}$ allows for the equilibrium rate to be not exogenous, although the estimated parameter is very small and insignificantly different from zero, as will be seen in Section 4. Equations (2) and (3) are identical to those used in Bomfim (2003), although our model substitutes the shadow rate for a spot rate in Bomfim's model and is more generalized by allowing $\sigma_{\theta x}$ to be nonzero.

The market prices of risk corresponding to the two innovations, $\lambda_{\theta,t}$ and $\lambda_{x,t}$, are assumed to be affine functions of the equilibrium rate and the shadow rate, respectively[4]:

$$\lambda_{\theta,t} = \lambda_{\theta 0} + \lambda_{\theta 1}\theta_t, \tag{4}$$

$$\lambda_{x,t} = \lambda_{x0} + \lambda_{x1}x_t. \tag{5}$$

The model consisting of equations (2)-(5) can be rewritten in a vector form:

$$d\mathbf{z}_t = \mathbf{K}(\boldsymbol{\theta} - \mathbf{z}_t)dt + \boldsymbol{\Sigma} d\mathbf{W}_t, \tag{6}$$

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1 \mathbf{z}_t, \tag{7}$$

where

[4] Duffee (2002) proposes essentially affine models, in which the market prices of risk are affine functions of the factors as equations (4) and (5). However, our model is not an affine model, since Black's models are nonlinear.

$$\mathbf{z}_t = \begin{pmatrix} \theta_t \\ x_t \end{pmatrix}, \ \boldsymbol{\theta} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} \kappa_\theta & 0 \\ -\kappa_x & \kappa_x \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_\theta & \sigma_{\theta x} \\ 0 & \sigma_x \end{pmatrix}, \ \mathbf{W}_t = \begin{pmatrix} W_{\theta,t} \\ W_{x,t} \end{pmatrix},$$

$$\boldsymbol{\lambda}_0 = \begin{pmatrix} \lambda_{\theta 0} \\ \lambda_{x0} \end{pmatrix}, \text{ and } \boldsymbol{\lambda}_1 = \begin{pmatrix} \lambda_{\theta 1} & 0 \\ 0 & \lambda_{x1} \end{pmatrix}.$$

Under the no-arbitrage assumption, the T-year nominal discount bond price can be expressed as

$$\exp\{-T \cdot \hat{i}^{(T)}(\mathbf{z}_t)\} = \tilde{E}_t[\exp\{-\int_t^{t+T} i_s ds\}], \qquad (8)$$

where $\hat{i}^{(T)}(\mathbf{z}_t)$ is the model-implied continuously compounded T-year nominal discount yield, and $\tilde{E}_t[\cdot]$ denotes the conditional expectation under the risk-neutral measure. The vector Ornstein-Uhlenbeck process (6) can be rewritten as

$$d\mathbf{z}_t = \tilde{\mathbf{K}}(\tilde{\boldsymbol{\theta}} - \mathbf{z}_t)dt + \boldsymbol{\Sigma} d\tilde{\mathbf{W}}_t, \qquad (9)$$

where $\tilde{\mathbf{K}}\tilde{\boldsymbol{\theta}} = \mathbf{K}\boldsymbol{\theta} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0$, $\tilde{\mathbf{K}} = \mathbf{K} + \boldsymbol{\Sigma}\boldsymbol{\lambda}_1$, $d\tilde{\mathbf{W}}_t = d\mathbf{W}_t + \boldsymbol{\lambda}_t dt$, and $\tilde{\mathbf{W}}_t$ consists of mutually uncorrelated standard Brownian motions under the risk-neutral measure. Given the parameter values and the factor vector $\mathbf{z}_t$, equation (8) can be solved to obtain $\hat{i}^{(T)}(\mathbf{z}_t)$ by applying a lattice method.[5]

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[5] See, for example, Amin and Bodurtha (1995) for lattice methods.

3. Estimation Method with Survey Data

This section describes the data and estimation method. The collateralized overnight call rate is used as the proxy for the spot nominal rate.[6] We also use 0.5-, 2-, 5-, and 10-year maturity zero-coupon yield data.[7] We use end-of-month data over the period from July 1996 to March 2006 to avoid a possible structural break in 1995, as argued by Miyao (2000). To overcome the short-sample problem, we use survey data as an additional source of information, as will be discussed below.

To estimate our continuous-time model with the monthly data, the model should be discretized. The discretized process of the factors is written as the following VAR(1) form:

$$\mathbf{z}_t = (\mathbf{I} - \mathbf{\Phi})\boldsymbol{\theta} + \mathbf{\Phi}\mathbf{z}_{t-1} + \mathbf{\Sigma}_\eta \boldsymbol{\eta}_t \, , \tag{10}$$

where $\mathbf{\Phi} = e^{-\mathbf{K}/12}$, $\mathbf{\Sigma}_\eta \mathbf{\Sigma}_\eta ' = \int_0^{1/12} e^{-\mathbf{K}s} \mathbf{\Sigma}\mathbf{\Sigma}' e^{-\mathbf{K}'s} ds$, and $\boldsymbol{\eta}_t \sim N(\mathbf{0}, \mathbf{I})$ i.i.d.[8] The Kalman filter is employed to estimate the parameters and the factor processes. In using the Kalman filter, conditional linearization is needed. Equation (1) can be linearized as $i_t = 1_{\{x_{t|t-1} \geq 0\}} x_t$, where $1_{\{x_{t|t-1} \geq 0\}}$ equals one if the optimal forecast of x_t at $t-1$, $x_{t|t-1}$ or the second element of $\mathbf{z}_{t|t-1} = (\mathbf{I} - \mathbf{\Phi})\boldsymbol{\theta} + \mathbf{\Phi}\mathbf{z}_{t-1}$, is nonnegative, and zero otherwise. The

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6 The Bank of Japan actually uses the uncollateralized call rate as a policy instrument. However, we use the collateralized rate, which is related less with the credit conditions of the market participants and is more appropriate for considering the relationship with risk-free government bond yields.

7 The zero-coupon yield data are constructed using a method proposed by McCulloch (1990).

8 In this paper, the notation $e^{\mathbf{X}}$, where $\mathbf{X}$ is a square matrix, denotes the matrix exponential $e^{\mathbf{X}} = I + \mathbf{X} + \mathbf{X}^2/2 + \mathbf{X}^3/6 + \cdots$.

observed overnight (O/N) call rate $i_t^{\text{O/N}}$ is assumed to equal the spot rate with an error $\varepsilon_t^{\text{O/N}}$:

$$i_t^{\text{O/N}} = 1_{\{x_{t|t-1} \geq 0\}} x_t + \varepsilon_t^{\text{O/N}}. \tag{11}$$

The model-implied T-year yields $\hat{i}^{(T)}(\mathbf{z}_t)$ are also conditionally linearlized and the observed yields $i_t^{(T)}$ are assumed to equal the model-implied ones with errors $\varepsilon_t^{(T)}$:

$$i_t^{(T)} = \hat{i}^{(T)}(\mathbf{z}_{t|t-1}) + \hat{i}^{(T)}{}'(\mathbf{z}_{t|t-1}) \cdot (\mathbf{z}_t - \mathbf{z}_{t|t-1}) + \varepsilon_t^{(T)}, \tag{12}$$

for $T = 0.5$, 2, 5, and 10.[9] The differentiated model-implied yield $\hat{i}^{(T)}{}'(\mathbf{z}_{t|t-1})$ can be calculated using the lattice method. Note that $\hat{i}^{(T)}{}'(\mathbf{z}_{t|t-1})$ indicates the degree to which factors $\mathbf{z}_t$ influence the T-year yields, and is interpreted as time-varying factor loadings.

Following Kim and Orphanides (2005), we use survey data of 3-month interest rates as an additional source of information to overcome the short-sample problem. To be specific, we use monthly data on the 6-months-ahead forecasts of the 3-month money market rates based on the Quick Survey System (QSS).[10] The survey forecast began to rise in October 2005, much earlier than July 2006, when the BOJ terminated the ZIRP

[9] Duffee (1999) employs a similar Kalman filter with conditional linearization for his defaultable bond pricing model.

[10] The QSS began to survey professional forecasts including that of the 3-month rate on Negotiable Certificates of Deposit (NCD) in July 1996, and then changed to survey the forecast of TIBOR (Tokyo Interbank Offered Rate) instead of the NCD rate.

and began to raise the target rate.[11] Thus using the survey forecasts should help estimate the yield curve model, especially in a low interest rate environment. Let $i_t^{(0.25)*}$ denote the forecast for a survey conducted at t. This survey is assumed to equal the model-implied 6-months-ahead expectations of 3-month rates with a constant c and an error ε_t^{3M*}:

$$i_t^{(0.25)*} = c + \hat{i}^{(0.25)}(\mathbf{z}_{t+6|t-1}) + \hat{i}^{(0.25)}{}'(\mathbf{z}_{t+6|t-1}) \cdot (\mathbf{z}_{t+6|t} - \mathbf{z}_{t+6|t-1}) + \varepsilon_t^{3M*}, \qquad (13)$$

where $\mathbf{z}_{t+6|t-1}$ and $\mathbf{z}_{t+6|t}$ are the model-implied optimal forecasts of $\mathbf{z}_{t+6}$ at $t-1$ and t, which satisfy $\mathbf{z}_{t+6|t-1} = (\mathbf{I} - \mathbf{\Phi}^7)\boldsymbol{\theta} + \mathbf{\Phi}^7 \mathbf{z}_{t-1}$ and $\mathbf{z}_{t+6|t} = (\mathbf{I} - \mathbf{\Phi}^6)\boldsymbol{\theta} + \mathbf{\Phi}^6 \mathbf{z}_t$, respectively. The constant term c denotes a mean spread in the money markets added on the government bond yield.

To examine the relationship between macroeconomic variables and the equilibrium interest rate, we use the semiannual data on forecasts of average real GDP growth rate and CPI percent change between approximately the next five to ten years based on the Consensus Forecasts. These are interpreted as long-run expected output growth and inflation that are not influenced by temporary imbalances in the economy and denoted g_t^* and π_t^* respectively. These surveys are used to restrict the model as

$$\theta_t = \delta + \gamma g_t^* + \pi_t^*. \qquad (14)$$

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[11] When the BOJ terminated the QMEP in March 2006, it adopted a policy similar to the ZIRP. Since the new policy is not different from the ZIRP in the sense that the BOJ controls the call rate to be as low as possible, we do not distinguish between these two.

This equation is motivated by the standard optimal growth model, and δ and γ can be interpreted as the rate of time preference and the inverse of intertemporal elasticity of substitution in consumption, respectively.[12] Note that the equilibrium interest rate can be regarded as an observable factor in the sense that the equilibrium rate is up to the observable survey forecasts, although the surveys are observable only semiannually, and the parameters linking the equilibrium rate and the surveys, δ and γ, are freely estimated.

The measurement errors $(\varepsilon_t^{\mathrm{O/N}}, \varepsilon_t^{(0.5)}, \varepsilon_t^{(2)}, \varepsilon_t^{(5)}, \varepsilon_t^{(10)}, \varepsilon_t^{3M*})'$ are assumed to follow serially and mutually uncorrelated normal distributions. In sum, we estimate 13 parameters in the model: θ, κ_θ, κ_x, σ_θ, σ_x, $\sigma_{\theta x}$, $\lambda_{\theta 0}$, λ_{x0}, $\lambda_{\theta 1}$, λ_{x1}, c, δ, and γ, except for the standard deviations of the measurement errors. The maximum likelihood estimation method is applied in estimating the model.

4. Model Fit and Parameter Estimates

This section evaluates the fit of the model, and then discusses the estimated parameter values. Table 1 reports the standard deviations of the measurement errors, according to which the 10-year yield is estimated with the highest standard deviation, 24bps. Figure 1, in fact, shows that the model-implied 10-year yield tracks the actual

[12] See Laubach and Williams (2001), for instance. To derive equation (14), it is assumed that the utility of each generation is weighted according to its size so that the equilibrium interest rate is independent from the rate of population growth. Although the output growth in (14) should be measured on a per-capita basis in the standard model, our measure is subject to population growth. However, this difference is likely negligible, since Japan's population growth rate has been stable, at around zero percent, over the period of our sample.

yield well, although there are some exceptions. The most prominent one is the sharp decline in 10-year yield toward lower than one percent in 1998. The intractable sharp decline presumably reflects special events such as the "flight-to-quality" reaction, in which the government bonds issued by developed countries attracted an extremely high demand as safer assets from investors facing Russian virtual default. Figure 2 shows the average yield curve during 2003 when the yield curve was most flattened as a year, and suggests that the model successfully replicate the convex shape of yield curve with a flatter slope for short to middle maturities. This flatter slope was observed when the short rate was very close to zero and is difficult to replicate by conventional models. In sum, these results show that the model fits the data quite well.

Table 2 reports the parameter estimates with the standard errors. The estimate of the steady state level of spot rate θ is 2.7 percent, around which the equilibrium rate has fluctuated. The rate of mean reversion of the equilibrium rate toward its steady state level κ_θ is much higher than that of the shadow rate toward the equilibrium rate κ_x. This shows that the equilibrium rate is more mean-reverting or less persistent than the shadow rate. The persistent deviation of the shadow rate from the equilibrium rate is highly consistent with Japan's experience: The Japanese economy faced extensive and pervasive deflationary pressure, and to counteract it the BOJ has kept the short rate extremely low for long time. Although the volatility of the shadow rate innovation σ_x is lower than that of the equilibrium rate innovation σ_θ, the persistency enables the shadow rate to have a significant impact on long-term yields, as will be shown. Note that $\sigma_{\theta x}$ is very small and insignificantly different from zero. Thus the equilibrium rate is regarded as an exogenous variable.

The parameter corresponding to the rate of time preference δ is very close to and insignificantly different from zero. On the other hand, the parameter corresponding to the inverse of the elasticity of intertemporal substitutions is 0.84, and insignificantly different from one. This result suggests that, although the parameters are not estimated with high accuracy, it may not be unreasonable to roughly approximate the equilibrium interest rate with the long-run expected nominal output growth rate, or the potential output growth rate plus the perceived inflation target, as often argued by market participants in Japan.[13]

5. Equilibrium Rate, Shadow Rate, and Yield Curve

This section discusses the implications of the estimated model. First, we observe the estimated time-series of the equilibrium interest rate in subsection 5.1. Subsection 5.2 reviews the relationship between the shadow rate and the expected duration of the BOJ's ZIRP to examine how the policy influenced the yield curve when the policy rate was zero. Subsection 5.3 examines the factor loadings to see how each factor influences the yield curve. Finally, subsection 5.4 decomposes the 10-year yield into the average expected short rate and the term premium.

[13] According to a BOJ announcement in March 2006, the level of inflation rate that each Policy Board member understands as price stability from a medium- to long-term viewpoint, in the conduct of monetary policy ("an understanding of medium- to long-term price stability"), was discussed, and most Board members' median figures fell on both sides of one percent. Some market participants often refer to the understanding as a proxy for the implicit inflation target to roughly estimate the equilibrium interest rate.

5.1. Equilibrium Interest Rate

Figure 3 shows the estimated equilibrium rate, and confirms that the equilibrium rate has fluctuated around 2.7 percent as the parameter value of the steady state level reported in Table 2. The figure also illustrates that the equilibrium rate has been volatile, reflecting the fluctuations in the long-run expected output growth and inflation shown in Figure 4 for the last decade, in which the Japanese economy has experienced many adverse episodes such as serious deflationary pressure and a banking crisis.

Broadly speaking, the estimated equilibrium rate shows a downtrend until 2003 and an uptrend since then, as the 10-year yield shown in Figure 1, and the survey forecasts of GDP growth and CPI percent change between approximately the next five to ten years, as shown in Figure 4. The equilibrium rate declined to approximately 1 percent around April 2003, when the 10-year yield, the long-range forecast of GDP growth, and the forecast of CPI inflation declined to 0.6 percent, 1.2 percent, and 0.1 percent, respectively. In particular, the forecast of CPI inflation was surprisingly low, which suggests that the fear of a deflationary spiral was extremely serious in Japan.

5.2. Shadow Rate

Figure 5 compares the estimated shadow rate with the CPI inflation rate. The shadow rate had been basically negative under the ZIRP and the QMEP conducted by the BOJ. The figure shows a strikingly close link between the shadow rate and the CPI inflation. The shadow rate follows the CPI inflation with a lag of five months on average, at which the correlation reaches 0.71. As argued by Ichiue and Ueno (2006), this relationship can be interpreted as the monetary policy rule committed by the BOJ. That is, even when the short rate was bounded at zero, the BOJ successfully controlled the shadow rate with its commitment on the link between the policy duration and inflation. That is to say, due to the commitment under the QMEP, market participants believed that the BOJ would not raise the short rate until the CPI inflation reached at least zero percent.

Figure 6 shows the expected first hitting-time of the zero shadow rate, which is calculated with Monte Carlo simulations and could be interpreted as the expected termination date of the ZIRP. The vertical difference between the expected termination date and the 45 degree line in the figure is the expected policy duration at each time, and is negatively correlated with the shadow rate. Thus, the shadow rate reflects the expected duration of the ZIRP, which drives the yield curve shape in that a more negative shadow rate—and thus a longer duration of ZIRP—results in a flatter yield curve. The expected duration of the ZIRP reached its peaks, 2.2 to 2.3 years, when the shadow rate reached the minimums, -0.7 to -0.8 percent, in 2002-2003. Note that with our two-factor model, the expected duration is determined not only by the shadow rate but also by the equilibrium rate. For instance, a lower equilibrium rate shrinks the spread between the equilibrium rate and the shadow rate, which alleviates the upward pressure on the shadow rate and extends the time for the shadow rate to reach the zero.

5.3. Factor Loadings

This subsection examines how much the equilibrium rate and the shadow rate influence the yield curve. Panels (a) and (b) in Figure 7 show the factor loadings $\hat{i}^{(T)}{}'(\mathbf{z}_{t|t-1})$ for 6-month and 10-year yields, as representative shorter and longer yields, respectively.

Panel (a) shows that the factor loading on the equilibrium rate has been close to zero, which suggests that the equilibrium rate have an only marginal effect on the shorter maturity yields. The factor loading on the shadow rate was also close to zero under the QMEP in 2001-2006. These results show that the shorter yields were almost fixed and insensitive to both factors under the QMEP. The factor loading on the shadow rate was around one half under the ZIRP in 1999-2000. However, when the BOJ did not adopt the ZIRP or QMEP, the factor loading was close to one, which implies that the shadow rate shifted the shorter yields almost in parallel.

Panel (b) suggests that both factors influence longer maturity yields. The QMEP only slightly weakened the relationship between the factors and the longer yields. In the other periods, one percent increases in the equilibrium rate and the shadow rate raise the 10-year yield by around 0.5 percent and 0.7-0.8 percent, respectively. The strong link between the shadow rate and the longer yields is caused by the persistency of the shadow rate. That is, once the shadow rate is lowered, the lower rate tends to be kept for a long time. Such expectations have contributed to pulling down the yield curve even at longer maturities.

5.4. Term Premium

Figure 8 decomposes the 10-year yield into the expectations component, which is the average of expected future short rates, as suggested by the pure expectations hypothesis, and the term premium, which is calculated as the actual 10-year yield minus the expectations component. This figure shows the expectations component varies more widely and plays a more important role in driving the 10-year yield than the term premium does, even under the QMEP. This result suggests that the policy duration effect—in which the commitment on the duration of zero interest rate influences the yield curve through the expected short rates—worked well.

The sharp drop in term premium in 1998 supports our view that the sharp decline in 10-year yield in that period was affected by special events such as the Russian crisis, which led to the flight-to-quality responses and pulled down the term premium. Interestingly, the term premium has been on a downtrend since 2004, which is caused by the measurement error shown in Figure 1. This result may suggest that the Japanese government bond market have faced a decline in term premium that cannot be explained by the model, as experienced in the United States since 2004.[14]

6. Conclusion

Equilibrium interest rates are useful indicators for both monetary policy authorities and market players, and thus have been estimated by many studies in the United States. However, there are few studies which estimate the equilibrium rate in Japan, due mainly to the following two difficulties in the data. First, the Japanese economy has experienced a zero or low interest rate environment for the past decade. Second, as the literature suggests, there may be a structural break in the Japanese economy in the mid-1990s. Our study overcomes this challenge by using survey data to

[14] Long-term interest rates in the United States had remained low since 2004, even though the Fed had raised the policy target on the federal funds rate from 1 percent to 5.25 percent for 2004-06. The literature, such as Rudebusch, Sack, and Wu (2006), argues that this observation is probably due to the decline in term premium. Bernanke (2006) discusses possible explanations for the declined premium, such as rapid growth in high-saving countries on the Pacific Rim.

estimate a Black's model of interest rates as options with two factors, which are interpreted as the equilibrium rate and the shadow rate. The survey forecasts of the 3-month rates work as additional information to overcome the difficulties arising from the persistent low interest rate environment and the short sample since the mid-1990s. In addition, using the survey data of the long-run expected output growth and inflation enables us to examine the relationship between the macroeconomic variables and the equilibrium interest rate.

The main findings are summarized as follows. First, the equilibrium interest rate has been volatile, reflecting the fluctuations in the long-run expected output growth and inflation. The fluctuation in the equilibrium rate has played an important role in driving the yield curve. For instance, a large part of the decline in the 10-year yield in 2003 was caused by the decline in the equilibrium rate, which reflects a serious deflationary scare, as suggested by only 0.1 percent of the survey forecast of CPI percentage change between approximately the next five to ten years.

Second, the estimated rate of time preference is very small, while the estimated inverse of the elasticity of intertemporal substitutions is insignificantly different from one. Thus, it is not unreasonable to approximate the equilibrium rate roughly with the long-run nominal output growth rate, as often argued by market participants in Japan.

Third, even when the decline of the short rate was bounded, the BOJ successfully lowered the shadow rate with its commitments, by which the market participants believed that the BOJ would not raise the short rate, for instance, until inflation reached at least zero percent. The decline in the shadow rate results in a reduction in the longer-term yields, even under the QMEP.

Finally, although expected future short rates have mainly driven the 10-year yield, the term premium also played some roles. The sharp decline in the 10-year yield in 1998 was caused by the decline in the term premium, probably due to the "flight-to-quality" reaction after the Russia crisis. The recent steady downtrend in the term premium has suppressed the 10-year yield even after the BOJ began to remove the policy accommodation in 2006.

Although BOJ terminated the ZIRP and began to raise its target on the overnight call rate in 2006, the target is still very low. In addition, we need to use the time-series data even before the termination of the ZIRP to secure sufficient samples for reliable empirical exercises. Therefore, for empirical analysis using Japan's data, we must keep handling the zero interest rate for next few decades, and the methods proposed in this paper may contribute to future studies.

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Table 1
Fit of the Model

Spot	0.5Y	2Y	5Y	10Y	3M*
11	2	9	15	24	7

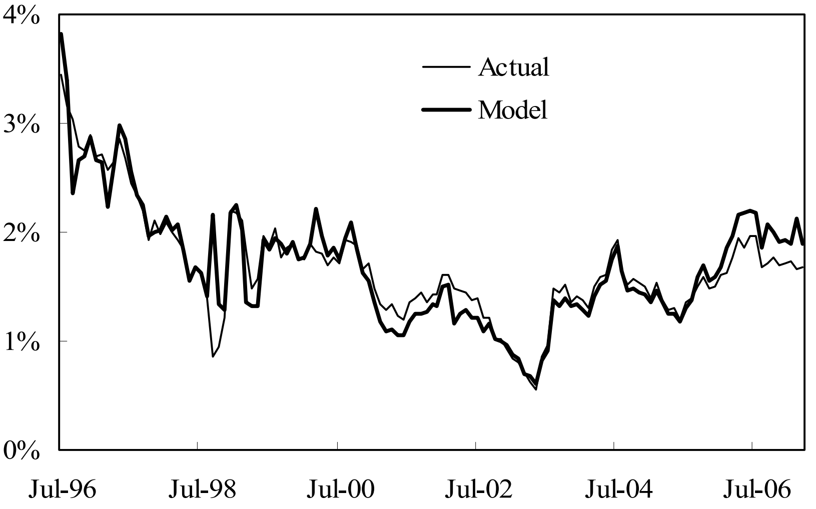
Note: This table reports the estimated standard deviations (basis points per year) of the measurement errors of spot rate (Spot), 0.5 to 10-year yields (0.5Y to 10Y), and 6-months-ahead expectations of the 3-month rates (3M*).

Table 2
Parameter Estimates

θ	κ_θ	κ_x	σ_θ	σ_x	$\sigma_{\theta x}$	
0.0273* (0.0089)	1.57* (0.31)	0.12* (0.02)	0.0200* (0.0036)	0.0043* (0.0005)	0.0001 (0.0008)	
$\lambda_{\theta 0}$	λ_{x0}	$\lambda_{\theta 1}$	λ_{x1}	c	δ	γ
-3.96* (0.52)	-0.27 (0.15)	3.0* (0.4)	-13.6* (0.4)	0.0012* (0.0003)	-0.0002 (0.0065)	0.84* (0.18)

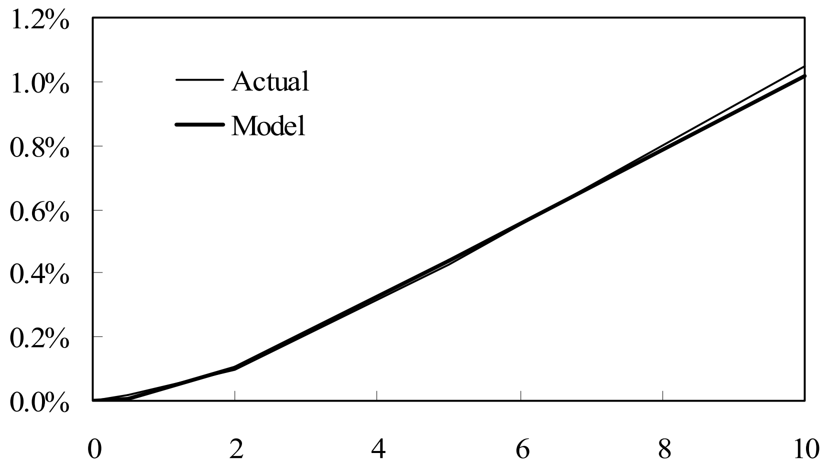
Note: Standard errors are in parentheses. The estimates with * are different from zero at the 5 percent significance level.

Figure 1
Model Fit for the 10-year Yield



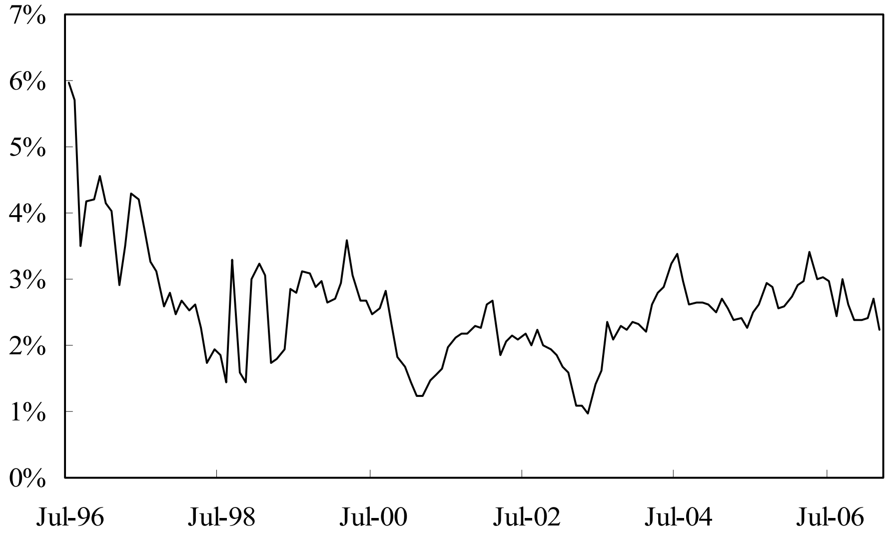
Note: The figure shows the actual (thin line) and model-implied (thick line) 10-year yields.

Figure 2
Average Yield Curve in 2003



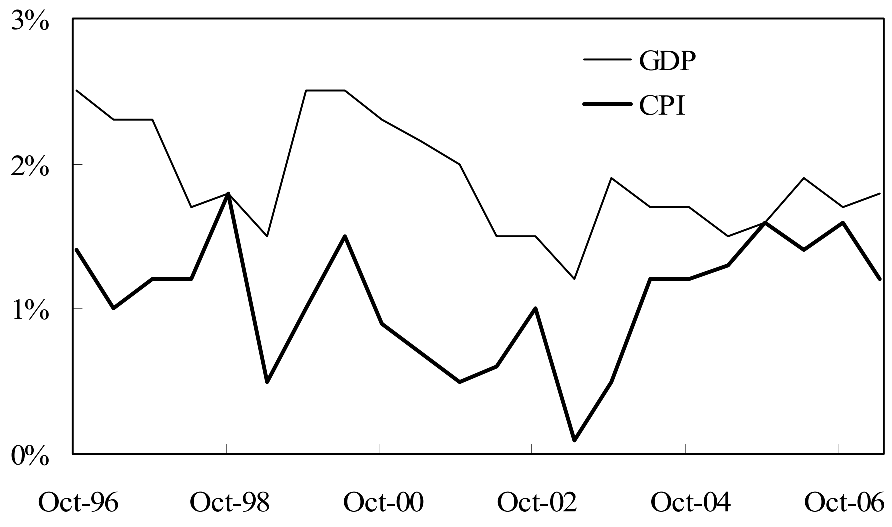
Note: This figure reports the averages of the actual and model-implied yield curves for 2003. The x-axis corresponds to the maturity (years).

Figure 3
Equilibrium Interest Rate



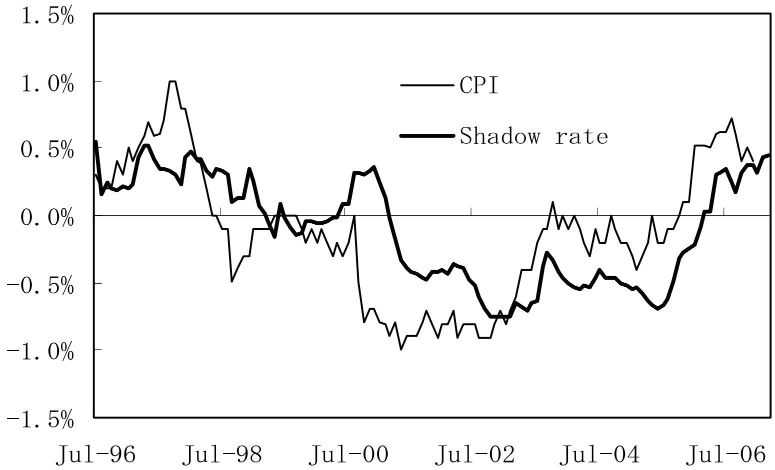
Note: This figure reports the estimated equilibrium interest rate.

Figure 4
Consensus Forecasts of Output Growth and Inflation



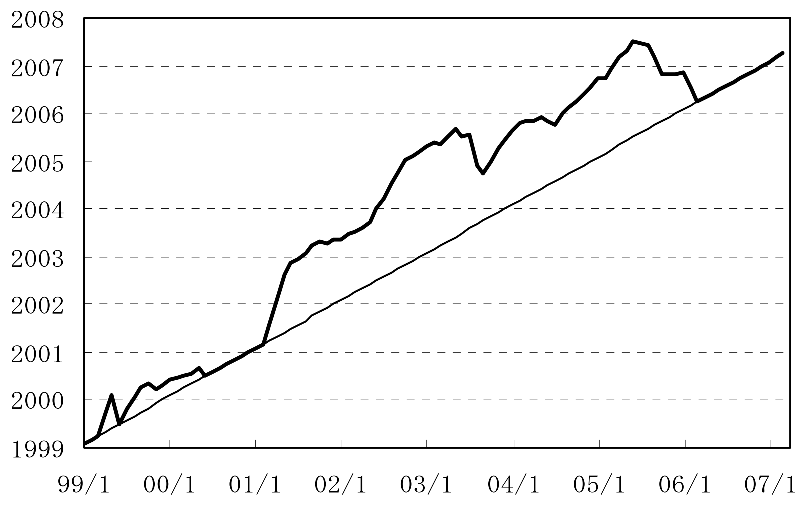
Note: This figure reports the Consensus Forecasts of real GDP growth (thin line) and CPI percent change between approximately the next five to ten years (thick line). Since the survey forecasts were not reported in April 2001, the lines are linearly interpolated. Note that the interpolated numbers are not used for our estimation.

Figure 5
Shadow Rate and Inflation Rate



Note: The figure compares the estimated shadow rate (thick line), and the year-on-year percentage increase of CPI (2000 base, excluding fresh food, the effect of consumption tax change adjusted; thin line).

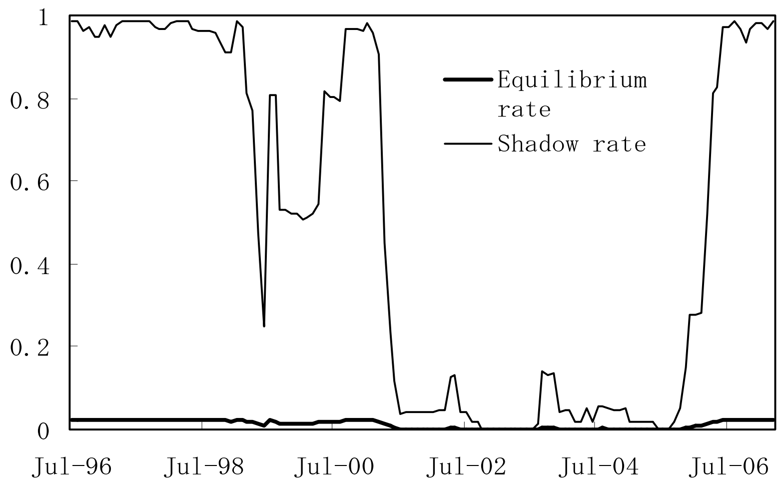
Figure 6
Expected Termination Date of the Zero Interest Rate Policy



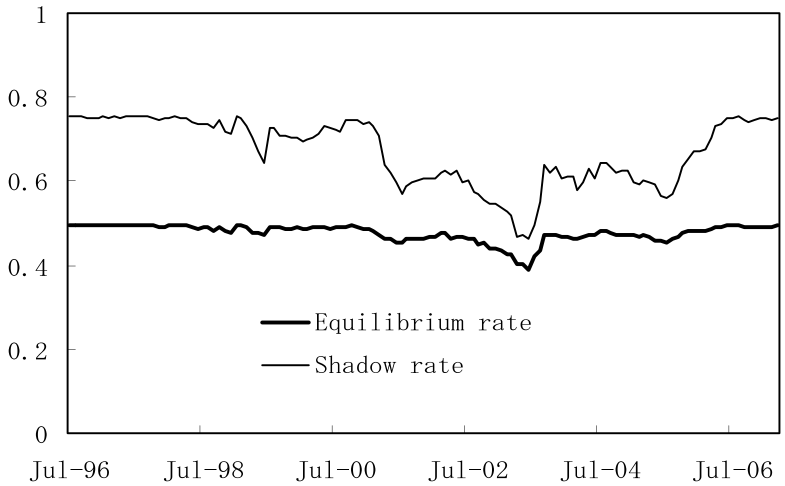
Note: This figure reports the estimation results of expected first hitting-time (years) of the zero shadow rate.

Figure 7
The Factor Loadings

(a) 6-month yield

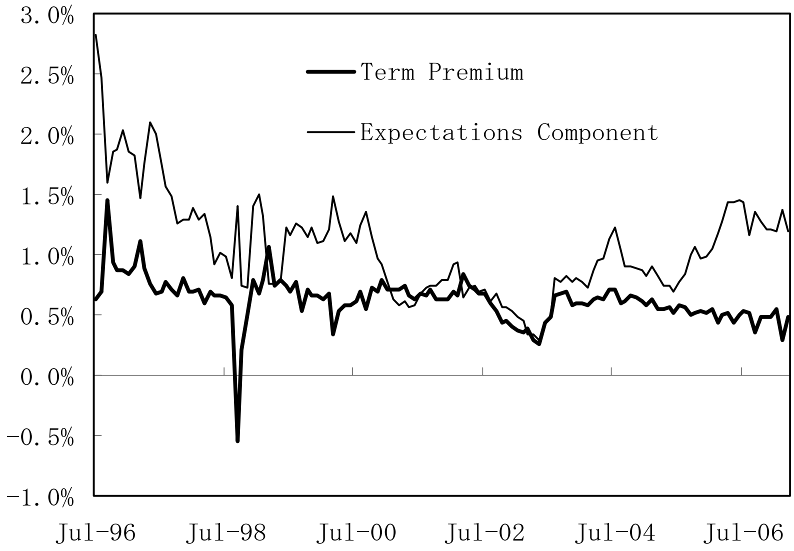


(b) 10-year yield



Note: Panel (a) and (b) report the factor loadings of the 6-month yield and 10-year yield, respectively. The thick and thin lines correspond to the factor loadings on the equilibrium interest rate and the shadow rate, respectively.

Figure 8
Decomposition of the 10-year Yield



Note: This figure reports the expectations component (thin line), which is the average of expected future short rates as suggested by the expectations hypothesis, and the term premium (thick line), which is calculated as the actual 10-year yield minus the expectations component.