Bank of Japan Working Paper Series

Asset correlation for credit risk analysis

―Empirical study of default data for Japanese companies―

Takashi Hashimoto*

Bank of Japan
2-1-1 Nihonbashi-Hongokucho, Chuo-ku, Tokyo 103-8660, Japan

No.09-E-3
August 2009

* Financial Systems and Bank Examination Department(currently Personnel and Corporate Affairs Department)

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[Summary]

This paper estimates and discusses asset correlations using a Merton-type factor model, based on time-series data on active and default companies in Japan by industry, size, credit rating and region. The results are as follows. First, one common factor is not always adequate for the precise estimation of asset correlations. Second, asset correlation varies across industry, size, credit rating and region groups. Third, asset correlation is high for large companies and low for small companies when grouped by size. Finally, asset correlation is high for high and low credit-rated companies, and low for middle credit-rated companies, when grouped by credit rating.

* Financial Systems and Bank Examination Department (currently Personnel and Corporate Affairs Department).

Correspondent: Risk Assessment Section, Financial Systems and Bank Examination Department (E-mail: post.fsbe65ra@boj.or.jp).

This paper is the English-translated version of BOJ Working Paper Series No.08-J-10 issued on June 2008 (in Japanese).

I would like to thank Prof. Soichiro Moridaira, Mr. Jun Muranaga, Mr. Toshifumi Ikemori, Regional Banks Association of Japan and the staff of the Bank of Japan for their helpful comments. The views expressed here, as well as any remaining errors, belong to the author and should not be ascribed to the Bank of Japan or Financial Systems and Bank Examination Department.

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1. Introduction

Financial institutions need to manage their credit risks to make a profit. In Japan, many financial institutions use an internal rating system to control their loans. In addition, they evaluate the quality of their portfolio using the Expected Loss (EL) and the Unexpected Loss (UL). When financial institutions calculate the UL, they often use a Merton model that requires setting up the following parameters: Probability of Default (PD), Loss Given Default (LGD), Exposure at Default (EaD) and Asset Correlation (AC). AC represents asset correlation of exposures to multiple debtors when they default simultaneously. Following the introduction of Basel II, they have been required to estimate the PD, LGD and EaD. However, estimation of the AC, which is not required for estimations associated with Pillar I of Basel II, has not often been discussed, even though it substantially affects the estimates of the UL. Importantly, it is essential that financial institutions use the appropriate method and data to calculate the AC in order to compute the UL accurately.

Theoretically, we can define the asset correlation for each loan. Generally, however, it is difficult to estimate all asset correlations of each loan because we cannot observe the value of each loan in the market and because financial institutions usually lend money to a large number of companies (borrowers). Therefore, in their risk analysis, financial institutions sum up loans by each company. Specifically, when they estimate asset correlations, they often construct groups of companies using a certain rule and estimate the asset correlation for each group. This rule is a key factor in estimating asset correlations and is essential for calculating the UL correctly. In this paper, we present several types of rules used to construct groups and compare the results of the asset correlations obtained.

In our analysis, we use the Teikoku Data Bank’s Matrix Data (1985–2005) to estimate the asset correlations for each group, because these data provide historical default data for Japanese companies. Using this data, we calculate the default rate for each year and group.

The structure of the paper is as follows. Section 2 presents the Merton Model used in the analysis. Section 3 provides the estimations. Section 4 discusses the results and draws conclusions.

2. One-factor Merton model

We estimate asset correlations using default data on Japanese companies. We use a one-factor Merton model[1] as it is an important model to calculate credit risk, not only in Japan but also elsewhere, because of its inclusion in the calculation of the capital adequacy requirements in Basel II.

The one-factor Merton model describes a company's value with a systematic factor (a factor common to the values of several companies) and an idiosyncratic factor (a factor specific to the company's value). The asset value of a company is then the weighted sum of a common (systematic) factor and an individual (idiosyncratic) factor. For example, when the macroeconomic development can be regarded as the systematic factor, the asset value of the company can be explained by the macroeconomic development and the company's individual factor. When these two factors change over time, the asset value of the company also changes.

In the Merton model, the occurrence of default is regarded as the time when the company's value is below a certain threshold at the time of maturity.

In this paper, we classify companies into several groups using a number of criteria (industry, size, credit and region)[2], and calculate the asset correlation for each group. We refer to the model that sets a common systematic factor for all companies as the Single Index Model and the model that sets a different systematic factor for each group of companies as the Multi Index Model. In this analysis, we mainly use the Multi Index Model.

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[1] See Appendix 1 for details of the single-factor model. See footnote 22 in Appendix 1 for details of the multifactor model.

[2] For example, when we use "industry" as a criterion, we can make groups of "Manufacturing", "Construction", and "Service". As another example, when we use "company size" as a criterion, we can make groups of "Large companies", "Small companies" and "Personal companies" (see footnote 11 for details of the definition of company size).

(1) Single Index Model

In the Single Index Model, the value $Z_i(t)$ of company a_i belonging to group S_k is:

$$Z_i(t) = \sqrt{\rho_k} X(t) + \sqrt{1-\rho_k}\varepsilon_i(t)$$

$$0 \le \rho_k \le 1,\ a_i \in S_k,\ i = 1,2,...,n,\ k = 1,2,...,m$$

where time t ($t \ge 0$), n is the number of companies and m is the number of groups.

The value Z_i of company a_i is described by two independent random variables: a systematic factor $X(t)$ (the factor common to all companies) and an idiosyncratic factor $\varepsilon_i(t)$ (the factor specific to company a_i). The companies that belong to group S_k have the same asset correlation or ρ_k, and $\sqrt{\rho_k}$ indicates the sensitivity of the company's value $Z_i(t)$ to the systematic factor $X(t)$. We assume $X(t)$ and $\varepsilon_i(t)$ follow a standard normal distribution independently of each other, meaning that $X(t)$ and $\varepsilon_i(t)$ are i.i.d. (independent and identically distributed). Therefore, $Z_i(t)$ also follows a standard normal distribution because $Z_i(t)$ is a linear combination of $X(t)$ and $\varepsilon_i(t)$.

(2) Multi Index Model

In the Multi Index Model, the value $Z_i(t)$ of company a_i belonging to group S_k is:

$$Z_i(t) = \sqrt{\rho_k} X_k(t) + \sqrt{1-\rho_k}\varepsilon_i(t)$$

$$0 \le \rho_k \le 1,\ a_i \in S_k, i = 1,2,...,n,\ k = 1,2,...,m$$

at time t ($t \ge 0$). When we establish $Z_i(t)$ in the Single Index Model, we use the systematic factor $X(t)$, which takes the same value for all companies. Conversely, in the Multi Index Model, we use the systematic factor $X_k(t)$, which takes different values for each S_k [3].

3. Estimating the asset correlations

In this section, we estimate the asset correlations using the Teikoku Data Bank's Matrix Data for calculating default rate. This database tracks Japanese company data

[3] See Appendix 1(3) for details.

from 1985 to 2005 and currently covers about 1.2 million companies.

To start with, we confirm the necessity to employ the Multi Index Model to calculate the UL. We then estimate the asset correlations for four groups: (1) industry type, (2) company size, (3) credit rating (the Teikoku Data Bank Score[4]), and (4) region. We use these particular groupings because analysts often use these criteria in credit risk management and because several previous studies use the same criteria[5].

Figure 1 depicts the time series of the total number of companies[6] and the number of default companies[7] in the database.

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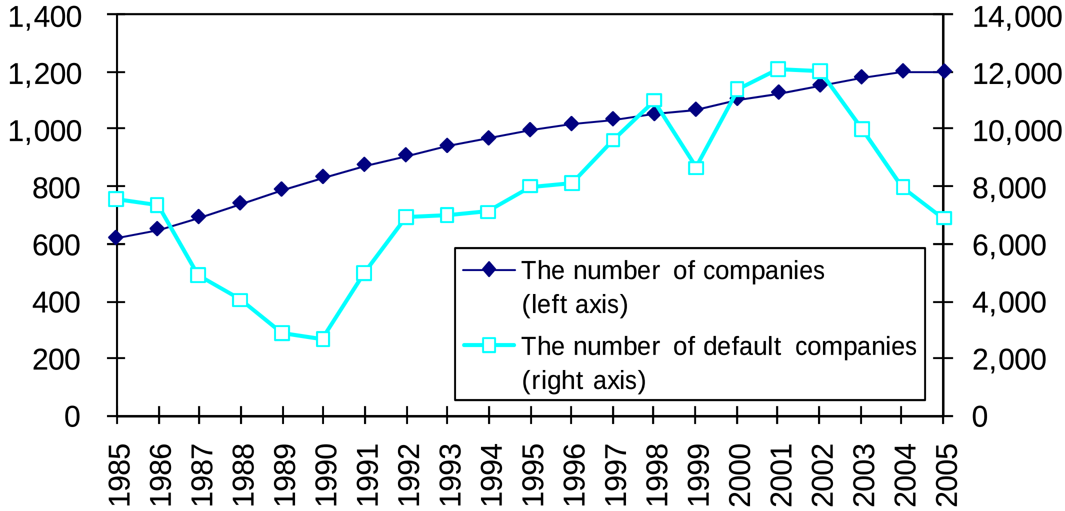
[4] The webpage of Teikoku Data Bank (in Japanese) describes the Score as follows: "The Teikoku Data Bank Score means how Teikoku Data Bank evaluates the company. The full score is 100. Teikoku Data Bank evaluates, as a third party, whether the company is well managed, is solvent and is able to deal with other companies safely."

[5] See Appendix 2 for details of previous studies.

[6] In this paper, we use company data included in the Teikoku Data Bank's Matrix Data, of which scores of the previous year end were given by the Teikoku Data Bank. We do not include company data for which Teikoku Data Bank shows "no score".

[7] In this paper, we define default as any of the following definitions of bankruptcy given by Teikoku Data Bank: (1) drawing unpaid notes twice and transactions with banks are suspended; (2) dissolution of the company (when the representative declares bankruptcy); (3) applying to the court for the application of the Corporate Rehabilitation Law; (4) applying to the court for the commencement of procedures based on the Civil Rehabilitation Law; (5) applying to the court for liquidation; and (6) applying to the court for the commencement of special liquidation.

[Figure 1] The number of companies and default companies in the database
(left axis: thousands of companies; right axis: number of companies)



(1) Reasons for using the Multi Index Model

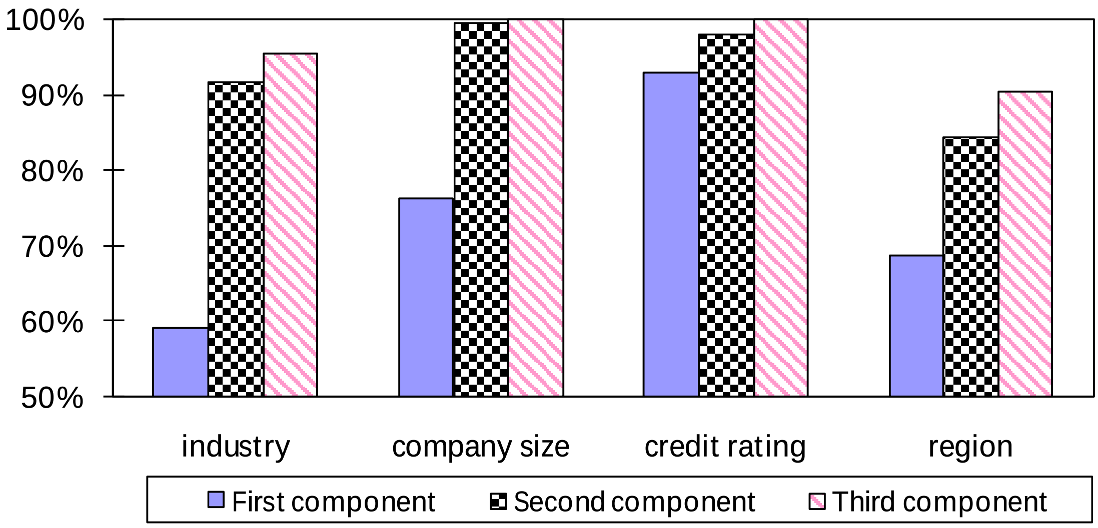
We use the Multi Index Model with Maximum Likelihood Estimation[8,9] when estimating the asset correlation. In this section, we describe why we use the Multi Index Model rather than the Single Index Model.

Figure 2 depicts the cumulative R-squared values of the principal component analysis using the time-series default rate data by industry type, company size, credit rating and region. Figure 2 shows that while the values of R-squared for the first component are low in all cases, the cumulative values of R-squared for the first and second components are more than 90 percent in most cases. This means that it is difficult to explain the changes in default rates only by the first component.

[8] See Appendix 3 for details of the Maximum Likelihood Estimation and Method of Moments and a comparison of the results.

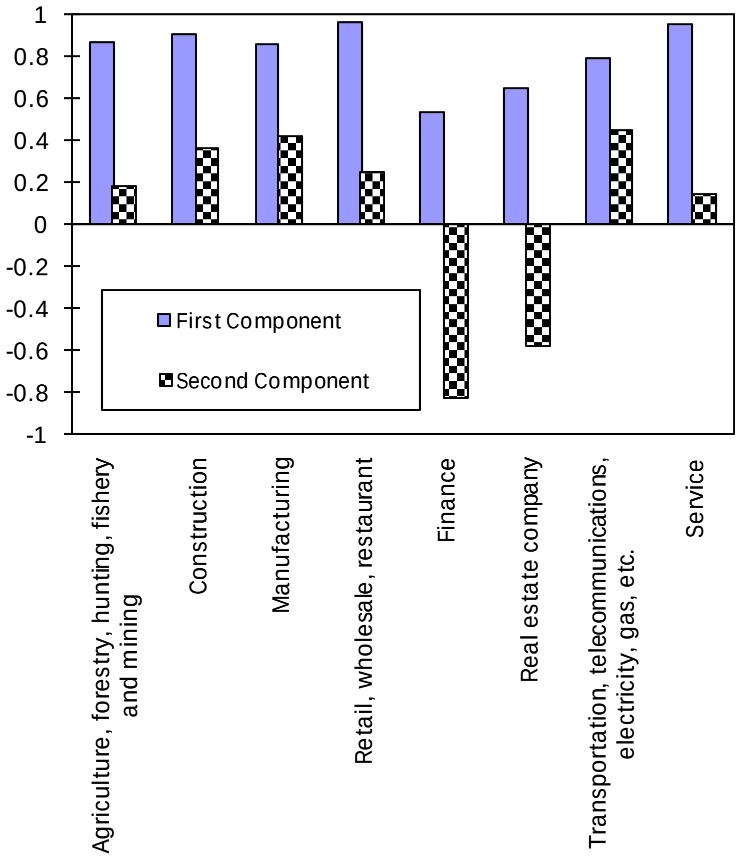
[9] We use MATLAB for the calculations. For the integral calculus in formula (15) in Appendix 3, we use quasi Monte Carlo integration by Halton sequence (the number of random variables is $2^{16} - 1 = 65,535$).

[Figure 2] Cumulative values of R-squared



This result shows that analysts must take into consideration both of the first and the second components. Figure 3 depicts the component loading for each type of industry.

[Figure 3] Component loading for each industry type



In Figure 3, the first component shows the common factor for all industry types but the second component does not. As shown, the first component displays the same sign for every industry type, while the second component has a different sign for each type of industry. In all types of industry, except finance and real estate, the common factor explains most of the change in default rate because the first component is larger than the second component. In contrast, in finance and real estate, the common factor cannot explain changes in the default rate because the second component is also large.

From this analysis, we conclude that a common factor alone cannot describe the asset value of companies because the values of R-squared for the time series of default rates indicate that finance and real estate display different results from other industries. Therefore, this paper proposes the following two-factor model, which contains not only an individual factor and a common factor but also a common group factor:

$$Z_i(t) = \sqrt{\alpha_k}\,X(t) + \sqrt{\beta_k}\,\delta_k(t) + \sqrt{1-\alpha_k^2-\beta_k^2}\,\varepsilon_i(t)$$

$$0 \le \rho_k \le 1,\ a_i \in S_k,\ i = 1,2,\dots,n,\ k = 1,2,\dots,m$$

This formula contains not only $X(t)$ (the factor common to all companies) and $\varepsilon_i(t)$ (the individual factor for company a_i), but also $\delta_k(t)$ (the factor common to group S_k that includes company a_i), where α_k and β_k are the asset correlations in a two-factor model. We set $\alpha_k = \rho_k \rho$ and $\beta_k = \rho_k \sqrt{1-\rho^2}$ and rewrite the above as:

$$\begin{aligned} Z_i(t) &= \sqrt{\rho_k}\sqrt{\rho}X(t) + \sqrt{\rho_k}\sqrt{1-\rho}\delta_k(t) + \sqrt{1-\rho_k}\,\varepsilon_i(t) \\ &= \sqrt{\rho_k}(\sqrt{\rho}X(t) + \sqrt{1-\rho}\delta_k(t)) + \sqrt{1-\rho_k}\,\varepsilon_i(t) \end{aligned}.$$

We then set $\sqrt{\rho}X(t) + \sqrt{1-\rho}\delta_k(t) = X_k(t)$,

$$Z_i(t) = \sqrt{\rho_k}\,X_k(t) + \sqrt{1-\rho_k}\,\varepsilon_i(t).$$

This is the same formula as in the Multi Index Model. For simplification, we apply in this paper the Multi Index Model that focuses on the estimation and analysis of the value of ρ_k.

(2) Grouping criteria and asset correlation

I. Industry

We group the Teikoku Data Bank's Matrix Data by industry type. Table 1 shows the categorization of the eight industry groups in this paper. Note that, because "agriculture", "forestry and hunting", "fishery" and "mining" include so few companies individually, we include these as a single industry, "agriculture, forest, hunting, fishery and mining". Note also that because the number of default companies in the "electricity, gas, water and heat supplier" industry is small, we combine it with the "transportation and telecommunications" industries into a composite group "transportation, telecommunications, electricity, gas, etc."[10]

[Table 1] Industry groups

Industry groups in this paper		Major industry groups in the Teikoku Data Bank's Matrix Data
1	Agriculture, forestry, hunting, fishery and mining	"Agriculture" + "Forestry and hunting" + "Fishery" + "Mining"
2	Construction	"Construction"
3	Manufacturing	"Manufacturing"
4	Retail wholesale, restaurant	"Retail and wholesale, restaurant"
5	Finance	"Finance"
6	Real estate	"Real estate"
7	Transportation, telecommunications, electricity, gas, etc.	"Electricity, gas, water and heat supplier" + "Transportation and telecommunications"
8	Service	"Service"

Figure 4 plots the transition of the default rate for each industry.

[Figure 4] Transition of default rate in each industry

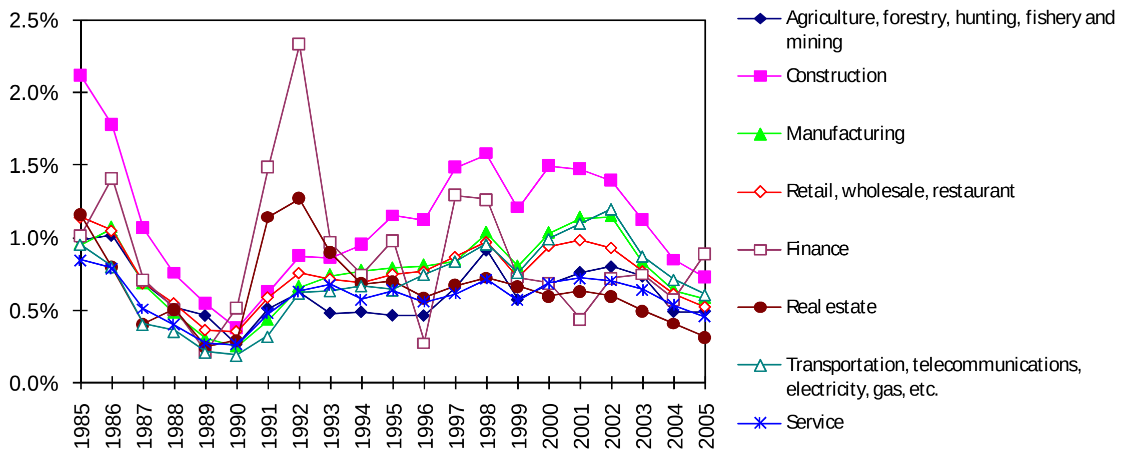
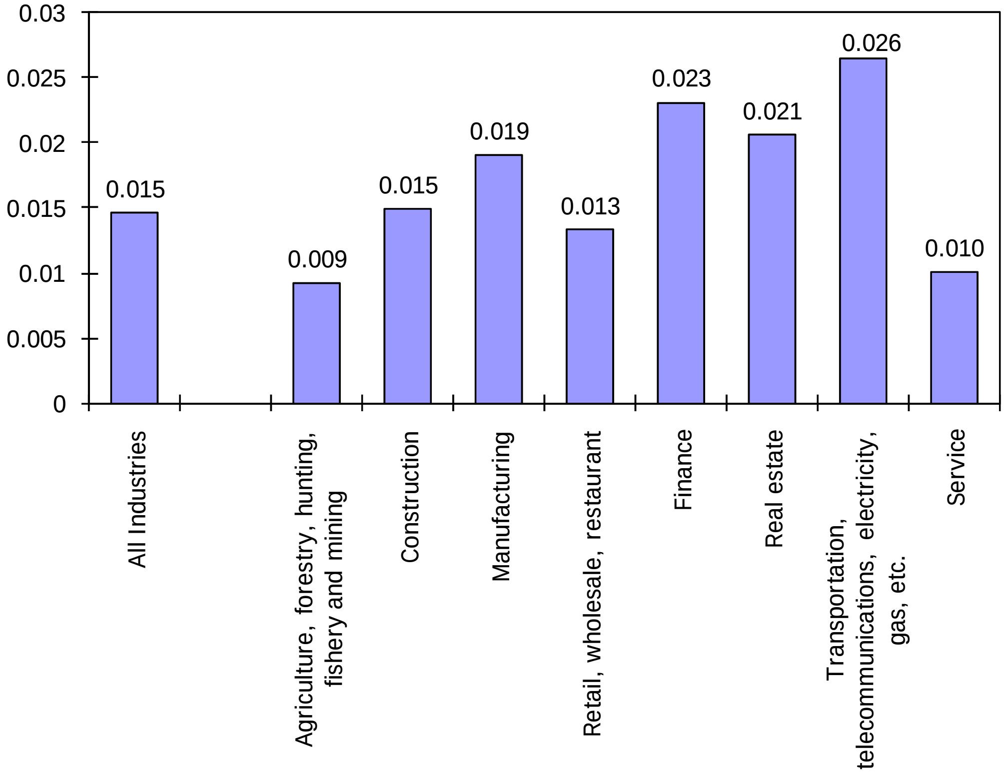


Figure 5 provides an estimate of the asset correlation for each industry. The data we use are the default status of the companies at the beginning of and during the term. As shown, the asset correlation differs across various industries.

[10] See appendix 4 for the number of companies and the number of default companies for each group.

[Figure 5] Asset correlation by industry



II. Company size

We group companies by size into three groups: "large and medium-sized companies", "small companies" and "personal companies"[11]. We combine large and

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[11] Our definitions of company size follow those given by Teikoku Data Bank:

Large companies	Capital of more than 1 billion yen (10 million US dollars) and more than 3,000 employees (more than 100 employees in wholesale; more than 50 employees in retail and service)
Medium-sized companies	Capital of more than 100 million yen (1 million US dollars) and more than 300 employees (in wholesale, capital of more than 30 million yen (0.3 million US dollars) and more than 100 employees; in retail and service, capital of more than 10 million yen (0.1 million US dollars) and more than 50 employees)
Small companies	Capital of less than 100 million yen (1 million US dollars), and less than 300 employees (in wholesale, capital of less than 30 million yen (0.3 million US dollars) and less than 100 employees; in retail and service, capital of less than 10 million yen (0.1 million US dollars) and less than 50 employees)
Personal companies	Companies that do not have legal corporate status

medium-sized companies into a single group because each includes few companies[12]. Figure 6 depicts the time series of the default rate by company size.

[Figure 6] Time series of the default rate by company size

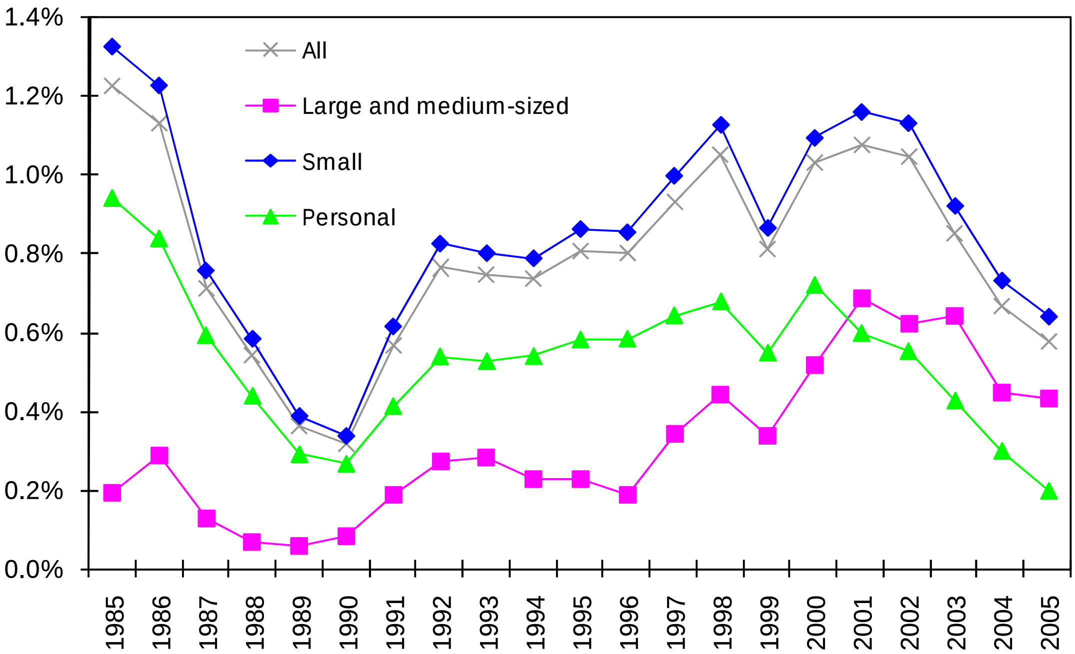


Figure 7 depicts the results of the estimation of asset correlation using the data on company size. The asset correlation of large and medium-sized companies is high and that of small companies and personal companies is low.

[Figure 7] Asset correlation by company size

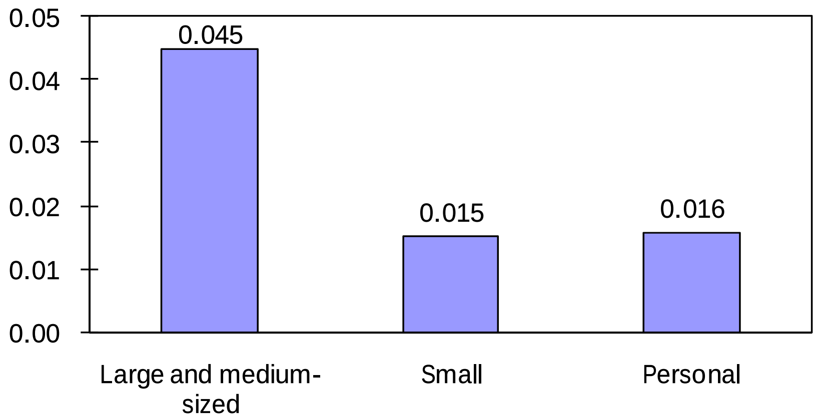
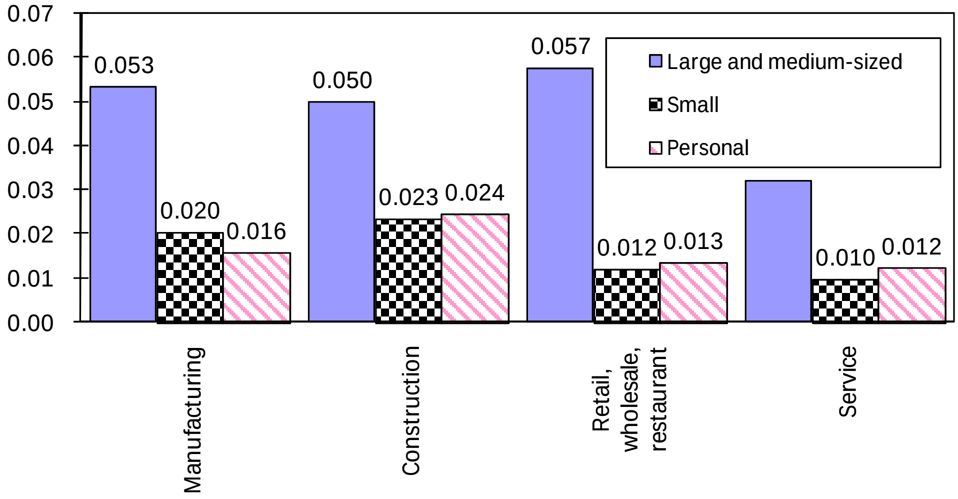


Figure 8 shows the asset correlation for each company size for four industries that

[12] See appendix 4 for the number of companies and the number of default companies for each group.

has a large number of companies. Similar to Figure 7, Figure 8 shows that asset correlation is high in large and medium-sized companies and low in small companies and personal companies.

[Figure 8] Asset correlation by company size for major industries



Literature survey[13] shows that, Düllmann and Scheule [2003], Lopez [2004] and Kitano [2007] argued that the smaller the size of the company, the lower the asset correlation[14]. Our findings are consistent with these previous studies. This could be attributed to the following hypotheses. First, the asset correlation of large companies is high because the performance of large companies is similar to the economic system—for example, the economic situation for a company in the particular region or industry to which the company belongs—and so large companies are affected by the systematic factor more than by the idiosyncratic factor. Second, the asset correlation of small companies and personal companies is lower because individual factors affect small companies more than the economic system does.

[13] See Appendix 2.

[14] Dietsch and Petey [2004] show that the shape of the asset correlation function is convex downward in terms of company size (the amount of sales). We explain this point further in Section 3, (2), III. Credit rating.

III. Credit rating

We group companies by credit rating using the Teikoku Data Bank Score (hereafter "Score")[15], on a scale with increments of 5. Figure 9 depicts the default rate for each scale.

[Figure 9] Time series of the default rate by credit rating

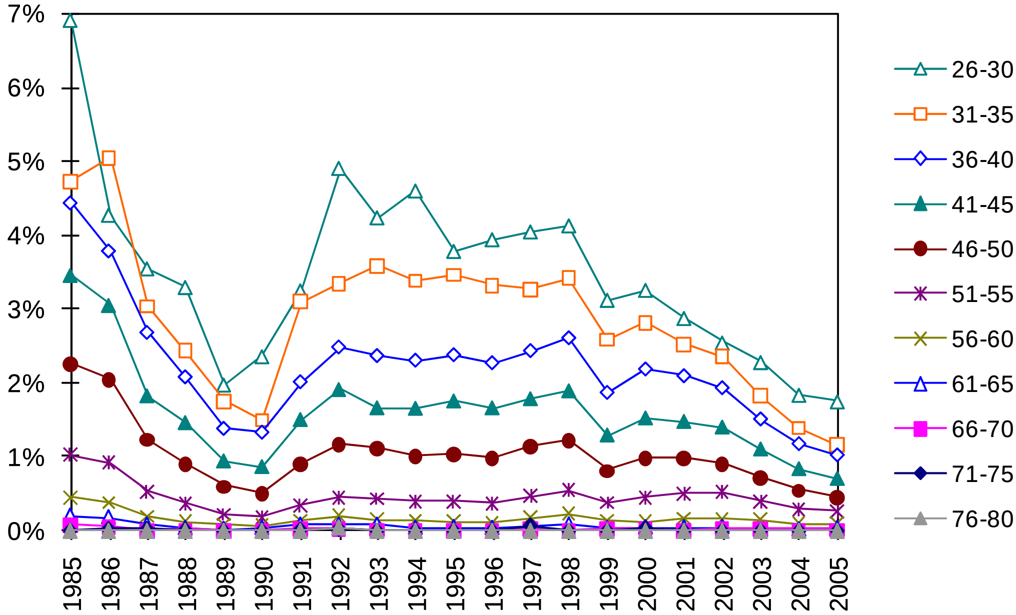
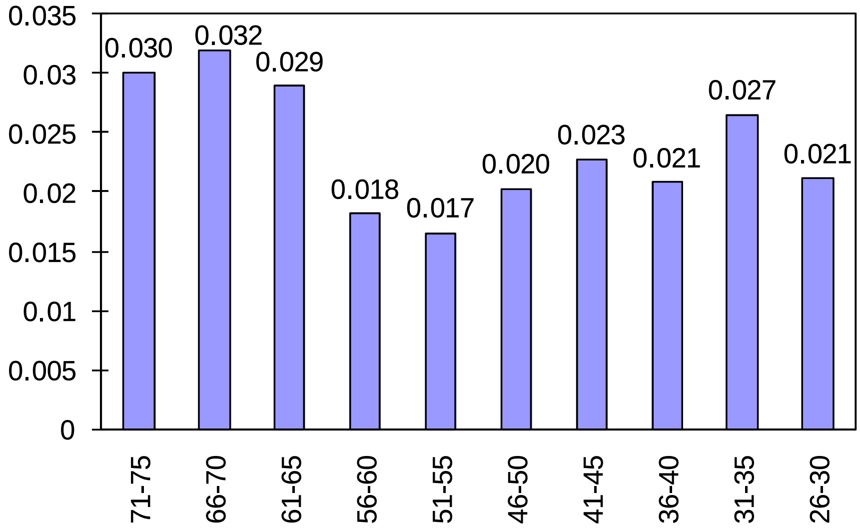


Figure 10 depicts the asset correlation for each scale of Score. Generally, asset correlation is high with high (Score is 75–61) and low (Score is 35–31) credit ratings and low with a middle credit rating (Score is 60–36).

[15] We did not include the data whose Score is lower than 25 or higher than 76 because the number of those data is small. See appendix 4 for the number of companies and the number of default companies for each group.

[Figure 10] Asset correlation for each Score range



In previous studies[16], Bluhm and Overbeck [2003] and Hamerle, Liebig and Rösch [2003] showed that asset correlation was high for companies with high and low credit ratings and low for companies with middle credit ratings. Dietsch and Petey [2004] found similar results for SMEs (small and medium-sized enterprises) in France, but not in Germany. Lopez [2004] also found lower asset correlations for lower credit-rated companies in the US, although companies in Europe and Japan did not clearly show this tendency[17]. Therefore, although not all results show the same trend, we can conclude from our results and those of previous studies that the shape of the asset correlation function can be convex downward.

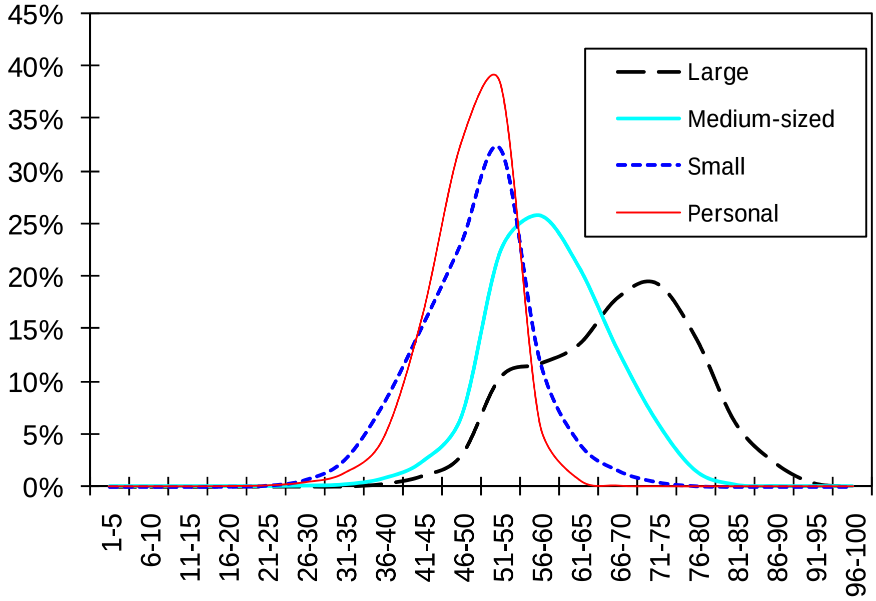
Figure 11 depicts the frequency distribution of the number of companies for each Score by company size. As shown, the smaller companies have higher rates of lower credit ratings. Therefore, there is a positive correlation between company size and credit rating.

[16] See Appendix 2.

[17] Düllmann and Scheule [2003] showed that with the exception of small sales companies, the lower the credit rating of a group, the larger the asset correlation. This is in contrast to other studies in this area.

[Figure 11] Relation between company size and Score

(vertical axis: frequency; horizontal axis: Score)



The shapes of the asset correlation functions are convex downward may relate to the finding of high asset correlations in large and medium-sized companies and low asset correlations in small and personal companies partly because of the positive correlation between credit rating and company size. If there is a positive correlation between credit rating and company size, the asset correlation in small companies could increase[18] because the asset correlation is high in companies with high and low credit ratings and low in companies with a middle credit rating. In fact, using French data, Dietsch and Petey [2004] concluded that the shapes of asset correlation functions are convex downward in the size (as measured by sales) of companies. Figures 7 and 8 illustrate that the asset correlation of small companies is lower than that for personal companies, but the difference is not significant. Although the asset correlation in small companies may be an important point of discussion for credit risk management, we leave this issue for future discussion work owing to the data availability[19].

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[18] This arises from one hypothesis that the smaller the company size, the more the company is influenced by the system because the assets of the company are not as diversified. Also this arises from the other hypothesis that the smaller the company size, the more the company is influenced by the change in bank's lending activity caused by the change in macroeconomic situation.

[19] The data used in this paper do not provide detailed groups for small and personal companies.

IV. Region

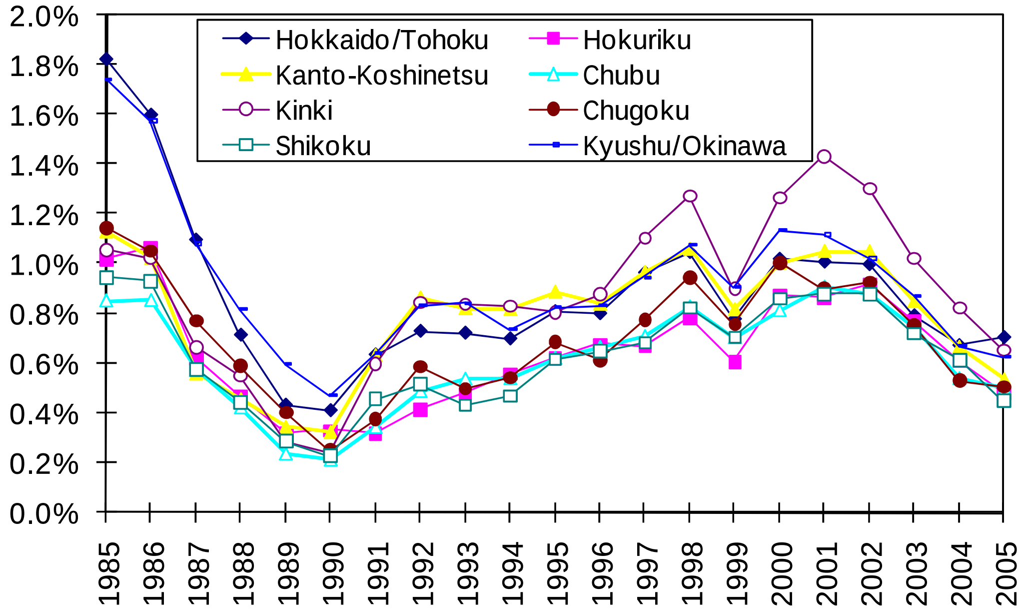
We group companies by (1) region and (2) prefecture (Table 2)[20].

[Table 2] Relation between region and prefecture in Japan

Region	Prefecture
Hokkaido/Tohoku	Hokkaido, Aomori, Iwate, Miyagi, Akita, Yamagata, Fukushima
Kanto–Koshinetsu	Ibaraki, Tochigi, Gunma, Saitama, Chiba, Tokyo, Kanagawa, Niigata, Yamanashi, Nagano
Hokuriku	Toyama, Ishikawa, Fukui
Chubu	Gifu, Shizuoka, Aichi, Mie
Kinki	Shiga, Kyoto, Osaka, Hyogo, Nara, Wakayama
Chugoku	Tottori, Shimane, Okayama, Hiroshima, Yamaguchi
Shikoku	Tokushima, Kagawa, Ehime, Kochi
Kyushu/Okinawa	Fukuoka, Saga, Nagasaki, Kumamoto, Oita, Miyazaki, Kagoshima, Okinawa

Figure 12 depicts the time series of the default rate in each region of Japan. Figure 12 shows that the default rate is: (1) high in Hokkaido/Tohoku and Kyushu/Okinawa before 1989, (2) high in Kinki from 1997 to 2004, and (3) low in Hokuriku, Chubu, and Shikoku throughout the whole period.

[Figure 12] Time series of default rates by district (all industries)



20 See appendix 4 for the number of companies and the number of default companies for each group.

Figure 13 depicts the time series of the default rate by prefecture. Figure 13 shows that the default rate: (1) is high in Okinawa before 1987, (2) is relatively high in Osaka from 1997 to 2004, and (3) does not differ much across prefectures in Hokuriku and Chubu.

[Figure 13] Time series of default rate in each prefecture (all industries)

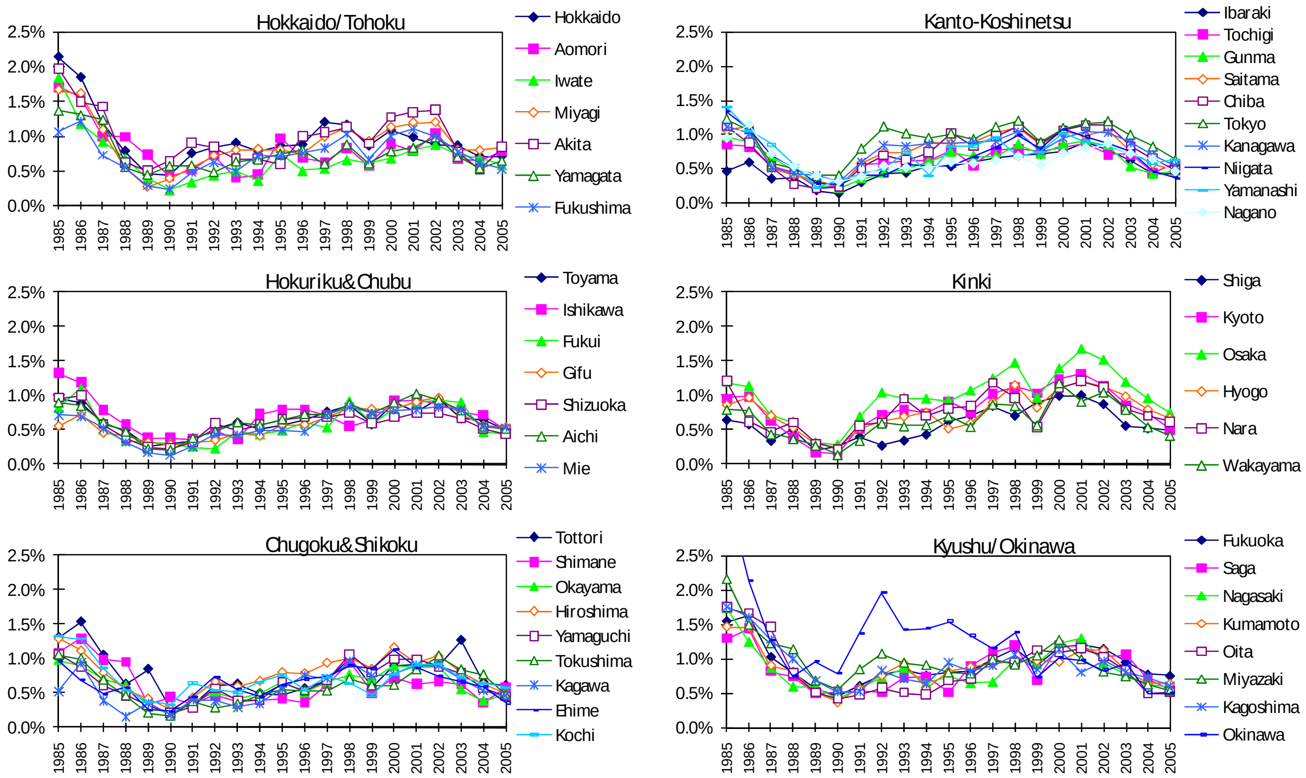


Figure 14 depicts the asset correlation in each prefecture and district. As shown, there are large differences in asset correlations among prefectures, with Kyoto (0.0340) recording 3.7 times of Yamagata (0.0092).

[Figure 14] Asset correlation by prefecture and district

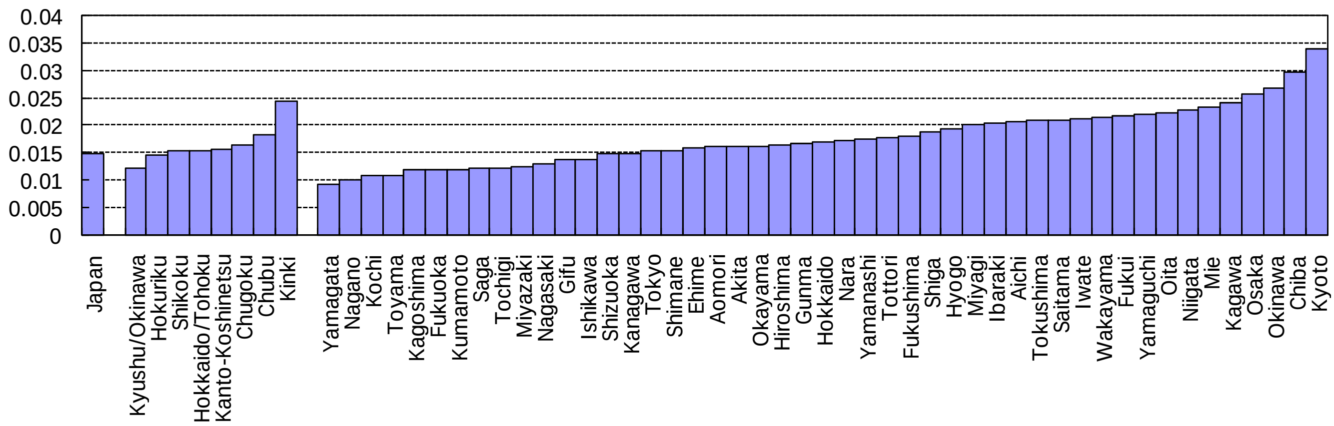
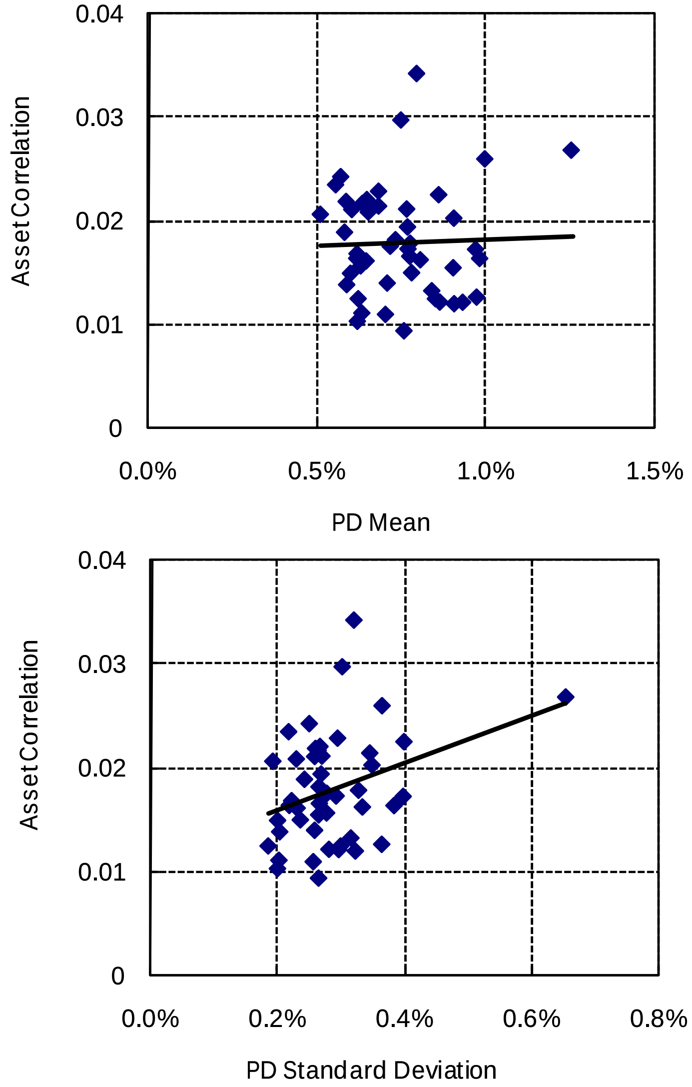


Figure 15 shows the relation between asset correlation of each prefecture and mean and standard deviation of PD.

[Figure 15] Relations between asset correlations in each prefecture and mean and standard deviation of PD



Applying simple linear regression analysis where asset correlation is the dependent variable and the standard deviation of PD is the independent variable, the estimated coefficient has a positive value of 2.266 and the t-value is 2.367, indicating significance at the 95% confidence level (Table 3). However, when we apply simple linear regression analysis where asset correlation is the dependent variable and the mean of PD is the independent variable, the estimated coefficient has a positive value of 0.1187, although the t-value of 0.227 indicates that it is not significant at any conventional level (Table 3).

[Table 3] Result of simple linear regression analysis
(the dependent variable is asset correlation)[21]

Independent variable	Estimated coefficient (t-value)	Intercept (t-value)	Coefficient of determination
PD mean	0.1187 (0.227)	0.01687 (4.234***)	0.00114
PD standard deviation	2.266 (2.367**)	0.01132 (4.017***)	0.1107

In conclusion, these results demonstrate that the relation between asset correlation and the standard deviation of the PD is positive. For example, the asset correlation in Okinawa, which has the largest standard deviation of the PD, is the third highest across all prefectures.

4. Conclusion

This paper analyzed the asset correlations of active and default Japanese companies. The data of these companies were grouped by industry type, company size, credit rating and region. The conclusions from this paper are as follows.

(1) The Multi Index Model was used because it was necessary to analyze at least both the first and the second components of the changes in default rate in order to explain changes in the default rate. Using principal component analysis, we cannot explain the cumulative R-squared of industry type, company size and region using only the first component, but can explain 90% of the R-squared using both the first and the second components.

(2) Asset correlations differ by industry, company size, credit rating and region. This means that it is not appropriate to use a common asset correlation in the credit portfolio.

(3) When the data are grouped by company size, asset correlation is high in large companies and low in small companies. Our hypothesis is that changes in the performance of large companies are similar to changes in the entire economic system, meaning, for example, a boom for all companies or higher performance in

[21] *** and ** indicate significance at the 99% and 95% confidence level, respectively.

the industry or region to which the company belongs. Therefore, the common factor easily affects the performance of large companies and so the asset correlation of these companies is high. On the other hand, the situation of the individual company rather than the economic system affects the performance of small companies. Therefore, asset correlations for small companies and personal companies are lower.

(4) Asset correlation is high in companies with high and low credit ratings and low in companies with middle credit ratings. In other words, the shapes of asset correlation functions are convex downward in credit ratings.

(5) The asset correlation among prefectures makes a great difference, with the maximum difference being 3.7 times. In addition, asset correlation and the standard deviation of the PD are positively related, and a prefecture where the volatility of the default rate is large tends to have a large asset correlation.

This analysis highlights the fact that asset correlation is an important subject for credit portfolio analysis in financial institutions. We expect that the findings of this paper will enable financial institutions in Japan to improve their credit risk management.

Appendix 1. One-factor model

The Merton-type factor model draws on Merton's [1974] view on the occurrence of default: we model the value of a company by changing stochastic behavior, and when the value of the company at maturity is below a given level (the default trigger), default occurs. This paper uses a one-factor Merton model[22].

(1) Basic formula

The basic formula of the one-factor Merton model is described by the following formula, where t ($t \geq 0$) is time and the asset value $Z_i(t)$ of company a_i is:

$$Z_i(t) = \sqrt{r_i}\, X(t) + \sqrt{1-r_i}\, \varepsilon_i(t), \qquad (1)$$
$$0 \leq r_i \leq 1,\ i = 1,2,...,n,$$

and n is the number of companies. The random variable of asset value $Z_i(t)$ is provided by two random variables: a common factor $X(t)$ that affects all companies and an idiosyncratic factor $\varepsilon_i(t)$ that affects only company a_i. $X(t)$ and $\varepsilon_i(t)$ are independent of each other and follow a standard normal distribution. Therefore, for the linear combination of these random variables on the right-hand side of (1), $Z_i(t)$ also follows a standard normal distribution. r_i is the asset correlation and $\sqrt{r_i}$ indicates the sensitivity of the asset value $Z_i(t)$ to the common factor $X(t)$.

A company a_i defaults when its asset value $Z_i(t)$ falls below the default trigger γ_i. Therefore, the probability of default PD_i of a_i describes the probability that $Z_i(t)$ is below γ_i:

$$\begin{aligned} PD_i &= \Pr\left(Z_i(t) < \gamma_i\right) \\ &= \Phi(\gamma_i) \end{aligned} \qquad (2)$$

---

[22] We can use a Multifactor Model in place of the one-factor Model. The Multifactor Model includes several common factors. For example, an F Factor Model includes F common factors, $Y_1(t), Y_2(t), \cdots, Y_F(t)$. In more concrete terms, the F Factor Model is:

$$Z_i(t) = \sqrt{r_{i,1}}Y_1(t) + \sqrt{r_{i,2}}Y_2(t) + \cdots + \sqrt{r_{i,F}}Y_F(t) + \sqrt{1-\sum_{j=1}^{F} r_{i,j}}\,\varepsilon_i(t) \qquad (a)$$

where $0 \leq r_{i,1} \leq 1$, $0 \leq r_{i,2} \leq 1$,..., $0 \leq r_{i,F} \leq 1$, $i = 1,2,...,n$ and $Y_1(t), Y_2(t), \cdots, Y_F(t)$, ε_i follow a standard normal distribution and are i.i.d.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-u^2/2) du$$

(2) Single Index Model

The Single Index Model is a type of one-factor Merton model. The Single Index Model imposes a single asset correlation for each group of companies decided by certain criteria, whereas the basic formula (1) imposes different asset correlations r_i for each company a_i.

In a set A composed of companies a_i $(i = 1, \cdots, n)$, the companies are grouped under certain criteria and the groups are described by $S^{(l)} = \{S_1^{(l)}, \cdots\cdots, S_{m_l}^{(l)}\}$. m_l shows the number of groups. Each $S^{(l)}$ is composed of a subset $S_k^{(l)}$ $(k = 1, \cdots, m_l)$, the element of which is the company a_i. l is the criterion used for making the group and k is the kind of group. For example, when l is "industry type", k is "manufacturing", "construction", and so on.

In any l, $S^{(l)}$ satisfies:

$$A = \bigcup_{k=1}^{m_l} S_k^{(l)} \quad , \quad S_i^{(l)} \cap S_j^{(l)} = \phi \, , \quad i \neq j \, , \quad S^{(l)} = \{S_1^{(l)}, \cdots\cdots, S_{m_l}^{(l)}\} \, .$$

To simplify the formula, we set only one l in $S^{(l)}$ (in other words, we decide the grouping criterion) and describe $S = \{S_1, \cdots, S_m\}$ instead of $S^{(l)}$ in the following parts.

We set the default trigger C_k and asset correlations ρ_k, ($r_i = r_j = \rho_k$, $a_i, a_j \in S_k$, $i \neq j$) of the company that belongs to the same group S_k as the same[23]. In other words, when $a_i \in S_k$, we can rewrite (1) and (2) as (3) and (4), respectively:

$$Z_i(t) = \sqrt{\rho_k} X(t) + \sqrt{1 - \rho_k} \varepsilon_i(t) \tag{3}$$

$$\begin{aligned} PD_i &= \Pr\left(Z_i(t) < C_k\right) \\ &= \Phi\left(C_k\right) \end{aligned} \tag{4}$$

We set these hypotheses because many credit risk managers set parameters not for

[23] We hypothesize that asset correlation ρ_k and default trigger C_k are the same in the same group. If we need to set several default triggers in the same group, we also need to set different triggers for different groups. This paper, like many others, hypothesizes that each group has the same default trigger C_k (see Appendix 2).

each company but for each group. It is possible to set different asset correlations r_i or default triggers γ_i in each company a_i when analysts can obtain detailed information on the asset value of company a_i or some alternative value such as its stock price, credit score, etc. However, in many cases, it is impossible or difficult to get these data. Therefore, analysts often set different asset correlations r_i or default triggers γ_i, not for each company a_i but rather for each group S_k.

(3) Multi Index Model

In the Single Index Model, all companies have the same common factor $X(t)$. However, in the Multi Index Model, companies in each group S_k follow a common factor $X_k(t)$ and each common factor $X_k(t)$ has a relation with each correlation. The Multi Index Model[24] replaces (3) above with the following:

$$Z_i(t) = \sqrt{\rho_k} X_k(t) + \sqrt{1-\rho_k} \varepsilon_i(t) \tag{5}$$

The difference between the Single Index Model (3) and Multi Index Model (5) is that the common factor is not a common $X(t)$ for all companies, but rather a common $X_k(t)$ for group S_k to which company a_i belongs.

The correlation between any pair of the common factors $X_k(t)$ and $X_l(t),\ k \neq l$ can, for example, be calculated using the following method (Bluhm and Overbeck [2003]).

The common factor $X_k(t)$ satisfies:

$$X_k(t) = \sqrt{\rho} X(t) + \sqrt{1-\rho} \delta_k(t). \tag{6}$$

Both $X(t)$ and $\delta_k(t)$ are independent and are independent of the idiosyncratic factor ε_i

[24] The coefficients of the formula in the Multi Index Model are different from those in the Multifactor Model described in footnote 22. However, we can describe the Multi Index Model using the Multifactor Model as follows. As we rewrote (1) to (3), we rewrite (a) in footnote 22 to:

$$Z_i(t) = \sqrt{r_{k,1}} Y_1(t) + \sqrt{r_{k,2}} Y_2(t) + \cdots + \sqrt{r_{k,F}} Y_F(t) + \sqrt{1 - \sum_{j=1}^{F} r_{k,j}} \varepsilon_i(t) \tag{b}$$

Let

$$\rho_k = \sum_{j=1}^{F} r_{k,j}$$
$$X_k(t) = \left(\sqrt{r_{k,1}} Y_1(t) + \sqrt{r_{k,2}} Y_2(t) + \cdots + \sqrt{r_{k,F}} Y_F(t) \right) \Big/ \sqrt{\rho_k} \tag{c}$$

in (5), then we have (b).

of company a_i.

Then, from (5) we get:

$$\begin{aligned} Z_i(t) &= \sqrt{\rho_k}(\sqrt{\rho}X(t) + \sqrt{1-\rho}\delta_k(t)) + \sqrt{1-\rho_k}\varepsilon_i(t) \\ &= \sqrt{\rho_k}\sqrt{\rho}X(t) + \sqrt{\rho_k}\sqrt{1-\rho}\delta_k(t) + \sqrt{1-\rho_k}\varepsilon_i(t) \end{aligned} . \quad (7)$$

In (7), the conditional default probability $p_k(X_k(t) \mid X_k(t) = x_k)$ ($p_k(x_k)$ in the following) of company a_i that belongs to group S_k under $X_k(t) = x_k$ is the probability that asset value $Z_i(t)$ falls below some threshold C_k.

From (5), $p_k(x_k)$ is:

$$\begin{aligned} p_k(x_k) &= \Pr(Z_i < C_k \mid X_k(t) = x_k) \\ &= \Pr(\sqrt{\rho_k}x_k + \sqrt{1-\rho_k}\varepsilon_i < C_k) \\ &= \Pr\left(\varepsilon_i < \frac{C_k - \sqrt{\rho_k}x_k}{\sqrt{1-\rho_k}}\right) \\ &= \Phi\left(\frac{C_k - \sqrt{\rho_k}x_k}{\sqrt{1-\rho_k}}\right). \end{aligned}$$

The covariance between $p_k(x_k)$ and $p_l(x_l)$ is:

$$\begin{aligned}\mathrm{Cov}[p_k(x_k), p_l(x_l)] &= \mathrm{E}[p_k(x_k)p_l(x_l)] - \bar{p}_k\bar{p}_l \\ &= \Phi_2(C_k, C_l \mid \rho\sqrt{\rho_k\rho_l}) - \Phi(C_k)\Phi(C_l)\end{aligned}, \qquad (8)^{25}$$

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[25] The proof of $\mathrm{E}[p_k(x_k)p_l(x_l)] = \Phi_2(C_k, C_l; \rho\sqrt{\rho_k\rho_l})$ is

$$\mathrm{E}[p_k(x_k)p_l(x_l)]$$
$$= \frac{1}{\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi\left(\frac{C_k-\sqrt{\rho_k}x_k}{\sqrt{1-\rho_k}}\right)\Phi\left(\frac{C_l-\sqrt{\rho_l}x_l}{\sqrt{1-\rho_l}}\right)\phi\left(\frac{x_k-\rho x_l}{\sqrt{1-\rho^2}}\right)\phi(x_l)\,dx_k\,dx_l \qquad \text{(i)}$$

When we change $y \equiv \frac{x_k - \rho x_l}{\sqrt{1-\rho^2}}$, we can rewrite part of (i) to:

$$\begin{aligned}&\frac{1}{\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\Phi\left(\frac{C_k-\sqrt{\rho_k}x_k}{\sqrt{1-\rho_k}}\right)\phi\left(\frac{x_k-\rho x_l}{\sqrt{1-\rho^2}}\right)dx_k \\ &= \int_{-\infty}^{\infty}\Phi\left(\frac{C_k-\sqrt{\rho_k}(\sqrt{1-\rho^2}\,y+\rho x_l)}{\sqrt{1-\rho_k}}\right)\phi(y)\,dy \\ &= \int_{-\infty}^{\infty}\Phi\left(\frac{\frac{C_k-\rho\sqrt{\rho_k}x_l}{\sqrt{1-\rho_k\rho^2}} - \sqrt{\frac{\rho_k-\rho_k\rho^2}{1-\rho_k\rho^2}}\,y}{\sqrt{\frac{1-\rho_k}{1-\rho_k\rho^2}}}\right)\phi(y)\,dy \\ &= \Phi\left(\frac{C_k-\rho\sqrt{\rho_k}x_l}{\sqrt{1-\rho_k\rho^2}}\right)\end{aligned}$$

Therefore, formula (i) changes to formula (ii):

$$\mathrm{E}[p_k(x_k)p_l(x_l)] = \int_{-\infty}^{\infty}\Phi\left(\frac{C_k-\rho\sqrt{\rho_k}x_l}{\sqrt{1-\rho_k\rho^2}}\right)\Phi\left(\frac{C_l-\sqrt{\rho_l}x_l}{\sqrt{1-\rho_l}}\right)\phi(x_l)dx_l \qquad \text{(ii)}$$

In addition, if we use:

$$\int_{-\infty}^{\infty}\Phi\left(\frac{a-cx}{\sqrt{1-c^2}}\right)\Phi\left(\frac{b-dx}{\sqrt{1-d^2}}\right)\phi(x)dx = \Phi_2(a, b \mid cd), \qquad (*)$$

then

$$\int_{-\infty}^{\infty}\Phi\left(\frac{C_k-\rho\sqrt{\rho_k}x_l}{\sqrt{1-\rho_k\rho^2}}\right)\Phi\left(\frac{C_l-\sqrt{\rho_l}x_l}{\sqrt{1-\rho_l}}\right)\phi(x_l)dx_l = \Phi_2(C_k, C_l \mid \rho\sqrt{\rho_k\rho_l}) \qquad \text{(ii)'}$$

Formula (i), like formula (iii), can be written as:

$$\mathrm{E}[p_k(x_k)p_l(x_l)] = \Phi_2(C_k, C_l \mid \rho\sqrt{\rho_k\rho_l}) \qquad \text{(iii)}$$

with a two-dimensional normal distribution function.

where $\overline{p}_k, \overline{p}_l$ are the nonconditional default probabilities.

Moreover, $\Phi_2(x_k, x_l \mid \rho)$ is the distribution function of a two-dimensional normal distribution such as:

$$\Phi_2(x_k, x_l \mid \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x_k} \int_{-\infty}^{x_l} \exp\left(-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right) dudv$$

$$\equiv \int_{-\infty}^{x_k} \int_{-\infty}^{x_l} \phi_2(u, v \mid \rho) dudv,$$

where $\phi_2(u, v \mid \rho)$ is the density function of the two-dimensional normal distribution.

When we set κ as the number of years of data, $\tilde{p}_{k,j}$ as the default probability of group S_k in year j, and $\tilde{\tilde{p}}_k$ as the mean of the default probability of the companies that belong to group S_k, covariance $\mathrm{Cov}[p_k(x_k), p_l(x_l)]$ is:

$$\mathrm{Cov}[p_k(x_k), p_l(x_l)] = \frac{1}{\kappa} \sum_{j=1}^{\kappa} (\tilde{p}_{k,j} - \tilde{\tilde{p}}_k)(\tilde{p}_{l,j} - \tilde{\tilde{p}}_l). \tag{9}$$

We can calculate the covariance on the left-hand side of formula (9) with time-series data on the default rate, and therefore can calculate ρ with both formula (8) and:

$$\frac{1}{\kappa} \sum_{j=1}^{\kappa} (\tilde{p}_{k,j} - \tilde{\tilde{p}}_k)(\tilde{p}_{l,j} - \tilde{\tilde{p}}_l) = \Phi_2(C_k, C_l \mid \rho\sqrt{\rho_k \rho_l}) - \Phi(C_k)\Phi(C_l). \tag{10}$$

If the value of ρ is not significantly different from 1, there is no difference in the common factor $X_k(t)$ across each group.

[the proof of formula (*)]

When we set $Y_1 = cX + \sqrt{1-c^2} Z_1$ $Y_2 = dX + \sqrt{1-d^2} Z_2$ $X, Z_1, Z_2 \sim N(0,1)$ i.i.d. ($-1 \le c, d \le 1$), the left-hand side of formula (*) shows the simultaneous probability, $\Pr(Y_1 < a, Y_2 < b)$. On the other hand, $\mathrm{Cov}[Y_1, Y_2] = cd$ and (Y_1, Y_2) clearly follow a two-dimensional normal distribution. This asset correlation is cd. Therefore, $\Pr(Y_1 < a, Y_2 < b) = \Phi_2(a, b \mid cd)$.

Appendix 2. Summary of previous studies of asset correlation estimation

Table 4 provides a summary of previous studies of asset correlation that use a Merton-type single-factor model with time-series default data.

[Table 4] Summary of previous studies of asset correlation using a Merton-type single-factor model[26]

Study	Data	Grouping criteria	Asset correlation
Carlos and Cespedes [2002]	Moody's (1970–2000)	–	About 0.1
Gordy and Heitfield [2002][27]	Moody's (1970–1998)	Credit rating	0.0551–0.1114
	S&P (1981–1997)		0.0494–0.0886
Hamerle, Liebig and Rösch [2003]	S&P (1982–1999)	Credit rating	0.0391–0.0695[28]
Bluhm and Overbeck [2003]	Moody's (1970–2001)	Credit rating	0.1177–0.4251
Düllmann and Scheule [2003]	Germany, 53,280 companies (1991–2000)	Credit rating and size	0.002–0.045[29]
Lopez [2004]	KMV CreditMonitor Database (6,909 US companies, 3,675 European companies, and 3,255 Japanese companies (2000)	Rating, size, and country	0.1000–0.5500
Dietsch and Petey [2004]	France, 440,000 companies (1995–2001)	Credit rating, size, and industry	0–0.1072
	Germany, 280,000 companies (1997–2001)		0–0.0652
Jakubik [2006]	Monthly default rate in Finland (1988/2–2004/1)	–	0.0152, 0.0166[30]
Kitano [2007]	Monthly default rate in Japan (data of Tokyo Shoko Research Ltd. and National Tax Agency 1982/7–2002/7)	Size	About 0.04–0.15[31]

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[26] Chernih, Vanduffel and Henrard [2006] have published a survey paper on asset correlation.

[27] This paper estimates the asset correlation for two cases: adding constraint conditions to the parameters and a no-constraint condition. Table 4 refers to the result of the no-constraint condition, which is basically the same estimation as ours. Moreover, this paper shows the result for the square root of the asset correlation that we define, so Table 4 shows the square times result of the paper.

[28] Table 4 shows the square times result of the paper.

[29] The paper uses two types of default rate: a default rate calculated by actual bankruptcy and one calculated by the reserve amount of lending. Table 4 shows the result of using a default rate calculated by actual bankruptcy using maximum likelihood methods.

[30] The threshold of asset value is whether the lenders are bankrupt. The paper uses two types of threshold: a fixed value and a function of GDP, etc. Table 4 refers to the results for a fixed value with a single-factor model. The difference in the two values in Table 4 is a data term. Furthermore, the paper calculates the asset correlation for each type of industry in the case that the threshold is defined by some function, but we do not describe this result in Table 4 because this method of calculating asset correlation is different from our chosen method.

[31] The paper deals with a two-factor model that is similar to formula (7) in this paper. However, as a special case of a two-factor model, a model that is the same as the Multi Index Model in this paper and called "Model One" in the paper is defined. Table 4 provides the results of this model.

Table 5 lists the asset correlation results of the above studies with the exception of Carlos and Cespedes [2002], Jakubik [2006] and Kitano [2007][32] by credit rating and size.

[Table 5] Asset correlation results of previous studies
for credit rating and size

・Gordy and Heitfield [2002]

(S&P credit rating)

A	BBB	BB	B	CCC
0.075	0.061	0.089	0.049	0.065

(Moody's credit rating)

A	Baa	Ba	B	Caa
0.055	0.084	0.111	0.067	0.063

・Hamerle, Liebig and Rösch [2003] (S&P credit rating)

BB	B	CCC
0.060	0.045	0.069

・Bluhm and Overbeck [2003] (Moody's credit rating)

Aa	A	Baa	Ba	B	Caa
0.3150	0.2289	0.1595	0.1300	0.1177	0.4251

・Düllmann and Scheule [2003] (credit rating, size)

Credit Hazard Rate (HR)		Sales		
		Small <5M euro	Medium 5–20M euro	Large >20M euro
A	$(0 < HR \leq 0.01)$	0.002	0.007	0.013
B	$(0.01 < HR \leq 0.015)$	0.010	0.011	0.016
C	$(0.015 < HR)$	0.005	0.016	0.045

・Lopez [2004] (S&P credit rating, size)

(US)

Rating	Asset scale		
	0–100M$	100–1000M$	1000M$–
AAA to BBB–	0.1375	0.1875	0.3250
BB+ to B–	0.1250	0.1875	0.2750
CCC+ to D	0.1250	0.1750	0.2250

[32] Kitano [2007] estimates the asset correlation for each size (amount of capital) of company. The results indicate that asset correlation is an increasing function of capital. See Kitano [2007] for details.

(Japan)

Rating	Asset scale		
	0–200M$	200–1000M$	1000M$–
AAA to BBB–	0.2250	0.2500	0.4250
BB+ to B–	0.2000	0.2500	0.4000
CCC+ to D	0.2000	0.2750	0.5550

(Europe)

Rating	Asset scale		
	0–100M$	100–1000M$	1000M$–
AAA to BBB–	0.1250	0.1250	0.2000
BB+ to B–	0.1250	0.1250	0.1750
CCC+ to D	0.1250	0.1250	0.1750

・Dietsch and Petey [2004] (credit rating, size)

(France)

Credit	Sales				Total SMEs
	Large firms >40M euro	SMEs 7–40M euro	SMEs 1–7M euro	SMEs <1M euro	
1 (high)	0.015	0.0279	0.0295	0.0079	0.0219
2	0	0.0156	0.0195	0.0012	0.0229
3	0.0439	0.0071	0.0061	0.0155	0.0231
4	0.0279	0.0057	0.0095	0.0134	0.0267
5	0.0277	0.0037	0.0098	0.0153	0.0151
6	0	0.0082	0.0147	0.0178	0.0199
7	0	0.0207	0.0208	0.0267	0.0298
8 (low)	0	0.1072	0.0279	0.0271	0.0307
Total	0.0221	0.0049	0.0097	0.0154	0.0128

(Germany)

Credit	Sales				Total SMEs
	Large firms >40M euro	SMEs 7–40M euro	SMEs 1–7M euro	SMEs <1M euro	
1 (high)	0.0121	0	0	0	0.0011
2	0.0251	0.0057	0.0133	0.0186	0.0129
3	0	0.0024	0.0129	0.0152	0.0119
4	0.0161	0.0652	0.0142	0.0221	0.0201
5	0.0075	0.0025	0.0202	0.0318	0.0259
6	0.0049	0.0025	0.0062	0.0121	0.0079
7	0.0169	0.0057	0.0197	0.0397	0.0275
8 (low)	0	0.0203	0.0262	0.0271	0.0259
Total	0.0145	0.0014	0.0079	0.0123	0.0093

Appendix 3. Estimation methods and differences in results

This appendix describes the estimation methods and analyzes the difference in results for each method. In this paper, the estimation methods are Maximum Likelihood Estimation and the Method of Moments. In the following, we use the Multi Index Model[33].

(1) Estimation methods

I. Method of Moments

The Method of Moments is a fitting method and can decide the best fitted asset correlation ρ_k with mean $\hat{\mu}_k$ and variance $\hat{\sigma}_k{}^2$ of the default rate in group S_k. The method of calculating the asset correlation using the Method of Moments is based on Gordy [2000].

The conditional default probability $p_k(X_k(t) \mid X_k(t) = x)$ ($p_k(x)$ hereafter) of a company a_i belonging to group S_k under $X_k(t) = x$ refers to asset value $Z_i(t)$ as below a certain threshold C_k and can be written as:

$$\begin{aligned} p_k(x) &= \Pr(Z_i(t) < C_k \mid X_k(t) = x) \\ &= \Pr(\sqrt{\rho_k} X_k(t) + \sqrt{1-\rho_k}\,\varepsilon_i(t) < C_k \mid X_k(t) = x) \\ &= \Pr\left(\varepsilon_i < \frac{C_k - \sqrt{\rho_k}\,x}{\sqrt{1-\rho_k}} \right) \\ &= \Phi\left(\frac{C_k - \sqrt{\rho_k}\,x}{\sqrt{1-\rho_k}} \right) \end{aligned} .$$

The situation of whether a company a_i ($a_i \in S_k$) in year t ($t = 1, 2, \ldots$) is in default or not is detailed below:

$$\tilde{H}_j^k(t) = \begin{cases} 1 & \text{default} \\ 0 & \text{non-default} \end{cases} \quad (j = 1, 2, \ldots, n_{k,t})$$

where $n_{k,t}$ is the number of companies.

Then, the default probability $\tilde{p}_k(t)$ observed in year t in group S_k is:

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[33] It is possible to present the same discussion with the Single Index Model in most cases when $X_k(t)$ is regarded as $X(t)$.

$$\tilde{p}_k(t) = \frac{1}{n_{k,t}} \sum_{j=1}^{n_{k,t}} \tilde{H}_j^k(t),$$

where $\bar{n}_k$, the average of time series $n_{k,t}$, is:

$$\bar{n}_k = \frac{1}{t} \sum_{\tau=1}^{t} n_{k,\tau},$$

and the mean and variance of $\tilde{p}_k(t)$ are, respectively, $\mathrm{E}_{\varepsilon_i}[\cdot]$ and $\mathrm{V}_{\varepsilon_i}[\cdot]$:

$$\begin{aligned} \mathrm{E}_{\varepsilon_i}[\tilde{p}_k(t) \mid X_k(t) = x] &\approx \frac{1}{\bar{n}_k} \sum_{j=1}^{n_{k,t}} \mathrm{E}[\tilde{H}_j^k(t)] \\ &= p_k(x) \end{aligned},$$

$$\begin{aligned} \mathrm{V}_{\varepsilon_i}[\tilde{p}_k(t) \mid X_k(t) = x] &\approx \frac{1}{\bar{n}_k^{\,2}} \sum_{j=1}^{n_{k,t}} \mathrm{V}[\tilde{H}_j^k(t)] \\ &= \frac{1}{\bar{n}_k^{\,2}} \left(\sum_{j=1}^{n_{k,t}} \mathrm{E}\left[\left(\tilde{H}_j^k(t)\right)^2\right] - \sum_{j=1}^{n_{k,t}} \mathrm{E}\left[\tilde{H}_j^k(t)\right]^2 \right) \\ &= \frac{1}{\bar{n}_k^{\,2}} \left(\sum_{j=1}^{n_{k,t}} \mathrm{E}\left[\tilde{H}_j^k(t)\right] - \sum_{j=1}^{n_{k,t}} \mathrm{E}\left[\tilde{H}_j^k(t)\right]^2 \right) \\ &= \frac{1}{\bar{n}_k^{\,2}} \left(\bar{n}_k p_k(x) - \bar{n}_k p_k(x)^2 \right) \\ &= \frac{p_k(x)(1 - p_k(x))}{\bar{n}_k} \end{aligned}.$$

The mean and variance of the actual default rate in each year, $\tilde{\mu}_k$ and $\tilde{\sigma}_k^{\,2}$, respectively, are given by:

$$\begin{aligned} \tilde{\mu}_k &= \mathrm{E}_{X_k,\varepsilon_i}[\tilde{p}_k(t)] \\ &= \mathrm{E}_{X_k}[\mathrm{E}_{\varepsilon_i}[\tilde{p}_k(t) \mid X_k(t) = x]], \\ &= \mathrm{E}_{X_k}[p_k(x)] \end{aligned}$$

$$\begin{aligned} \tilde{\sigma}_k^{\,2} &= \mathrm{V}_{X_k,\varepsilon_i}[\tilde{p}_k(t)] \\ &= \mathrm{V}_{X_k}[\mathrm{E}_{\varepsilon_i}[\tilde{p}_k(t) \mid X_k(t) = x]] + \mathrm{E}_{X_k}[\mathrm{V}_{\varepsilon_i}[\tilde{p}_k(t) \mid X_k(t) = x]]. \\ &= \mathrm{V}_{X_k}[p_k(x)] + \mathrm{E}_{X_k}\left[\frac{p_k(x)(1 - p_k(x))}{\bar{n}_k} \right] \end{aligned} \qquad (11)$$

The variance $\mathrm{V}_{X_k}[p_k(x)]$ used in the parameter calculation is:

$$\mathrm{V}_{X_k}[p_k(x)] = \frac{\mathrm{V}_{X_k,\varepsilon_i}[\tilde{p}_k(t)] - \frac{1}{\overline{n}_k}\mathrm{E}_{X_k}\left[p_k(x)\right]\left(1-\mathrm{E}_{X_k}\left[p_k(x)\right]\right)}{1-\frac{1}{\overline{n}_k}}.$$

$$= \frac{\overline{n}_k\tilde{\sigma}_k{}^2 - \tilde{\mu}_k + \tilde{\mu}_k{}^2}{\overline{n}_k - 1}$$

Therefore, C_k and ρ_k can be solved with the following simultaneous equations:

$$\tilde{\mu}_k = \Phi(C_k) \tag{12}$$

$$\frac{\overline{n}_k\tilde{\sigma}_k{}^2 - \tilde{\mu}_k + \tilde{\mu}_k{}^2}{\overline{n}_k - 1} = \int_{-\infty}^{+\infty}\left\{\Phi\left(\frac{C_k-\sqrt{\rho_k}\,x}{\sqrt{1-\rho_k}}\right)\right\}^2 d\Phi(x) - (\Phi(C_k))^2$$
$$= \Phi_2(C_k, C_k \mid \rho_k) - \Phi_2(C_k, C_k \mid 0) \qquad (13)[34,35]$$

We refer to the calculation method based on simultaneous equations of (12) and (13) as the Finite Method of Moments.

Furthermore, when $\overline{n}_k \to \infty$ formula (13) becomes:

$$\tilde{\sigma}_k{}^2 = \int_{-\infty}^{+\infty}\left\{\Phi\left(\frac{C_k-\sqrt{\rho_k}\,x}{\sqrt{1-\rho_k}}\right)\right\}^2 d\Phi(x) - (\Phi(C_k))^2$$
$$= \Phi_2(C_k, C_k \mid \rho_k) - \Phi_2(C_k, C_k \mid 0) \tag{14}$$

The parameter estimation can use formula (14) instead of formula (13). In this paper, we refer to the calculation method based on simultaneous equations of (12) and (14) as the Asymptotic Method of Moments.

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[34] Here, we use

$$\int_{-\infty}^{\infty}\Phi\left(\frac{a-cx}{\sqrt{1-c^2}}\right)\Phi\left(\frac{b-dx}{\sqrt{1-d^2}}\right)\phi(x)dx = \Phi_2\left(a,b \mid cd\right),$$

formula (*) in footnote 25. When we set $a = b = C_k$, $c = d = \sqrt{\rho_k}$, we can calculate the first term of formula (13), and when we set $a = b = C_k$, $c = d = 0$, we can calculate the second term of formula (13).

[35] During calculation, the left-hand side sometimes becomes negative. Therefore, this paper sets the threshold at zero and the value of the left-hand side at zero when the left-hand side is negative during the experiments in Figure 16 and Table 7.

II. Maximum Likelihood Estimation

The method of calculating asset correlation ρ_k by Maximum Likelihood Estimation is based on Gordy and Heitfield [2002]. We set the default probability $p_k(x)$ of group S_k, the number of companies in year t as $n_{k,t}$ and the number of default companies as $d_{k,t}$. The conditional likelihood function under $X_k(t) = x$ is:

$$L(d_{k,t} \mid X_k(t) = x) = \binom{n_{k,t}}{d_{k,t}} p_k(x)^{d_{k,t}} (1 - p_k(x))^{(n_{k,t} - d_{k,t})},$$

$$p_k(x) = \Phi\left(\frac{C_k - \sqrt{\rho_k}\,x}{\sqrt{1-\rho_k}}\right).$$

We set each year as $t = 1,2,...,T$ and each group to which companies belong as S_k $(k = 1,2,\ldots,m)$. The unconditional likelihood function is:

$$L = \prod_{t=1}^{T} \int_{-\infty}^{+\infty} \prod_{k=1}^{m} \binom{n_{k,t}}{d_{k,t}} p_k(x)^{d_{k,t}} (1 - p_k(x))^{(n_{k,t} - d_{k,t})} d\Phi(x). \qquad (15)$$

In the maximum likelihood method, C_k and ρ_k are calculated when L in the likelihood function (15) is at a maximum.

(2) Comparing the Method of Moments and Maximum Likelihood Estimation

In this section, we compare the Method of Moments and Maximum Likelihood Estimation. We set some virtual default data, calculate the parameters with the two methods, and then compare the results. First, we set the number of companies, the default rate, the asset correlations and the number of years artificially. Second, we construct some random variables using formula (4) and create the time series of virtual data for default companies for the value of the asset correlation. Third, we estimate the asset correlation again using this virtual data. Finally, we compare the result of the estimation with the original asset correlation that was set. Table 6 shows the parameter values set.

[Table 6] Parameter settings

Number of companies	32(≒ $10^{1.5}$) – 100,000(= 10^{5}), each $10^{0.25}$
Default rate	1%
Asset correlation	0.01, 0.10, 0.20
Number of years	10 (independent of one another)
Number of experiments	10,000

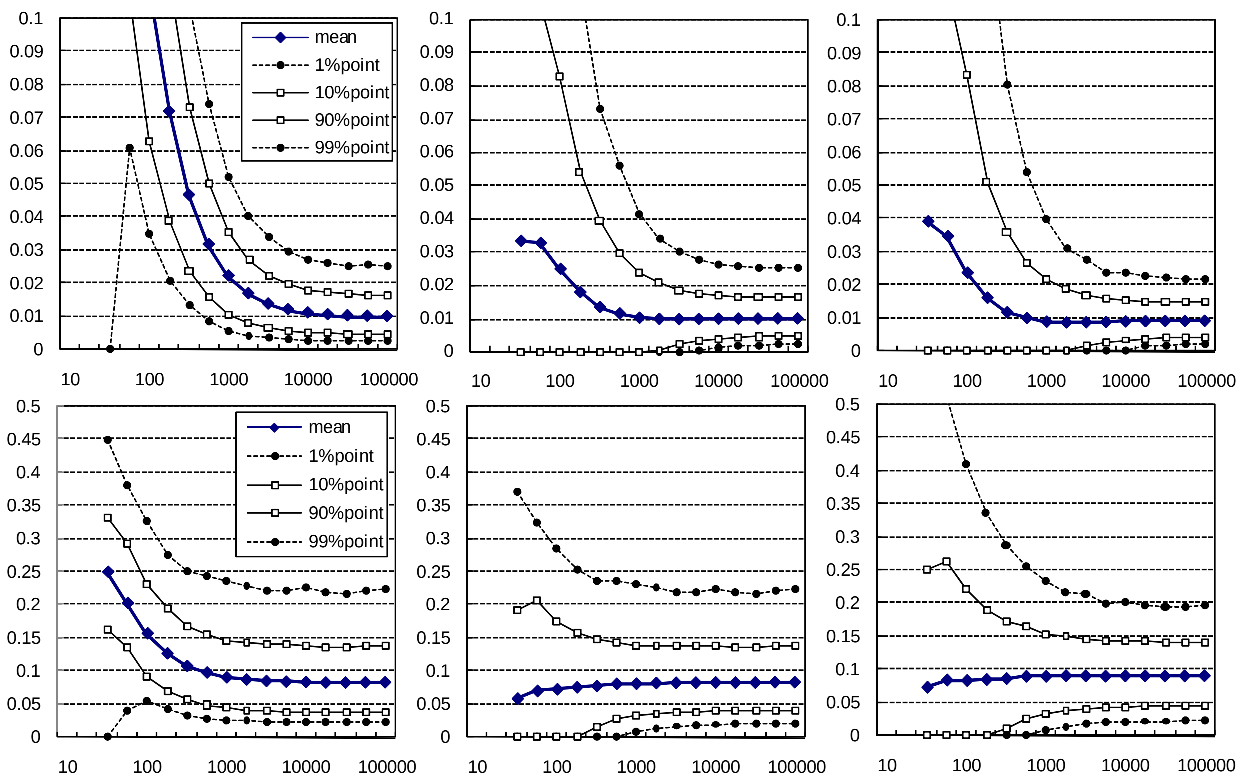
Using the parameters in Table 6, we create virtual data on the number of default and active companies using Monte Carlo methods and estimate the asset correlations for 10 years. This process recurs 10,000 times and we calculate the average, 99% point, 90% point, 10% point and 1% point. Figure 16 shows the results[36].

[Figure 16] Reestimated value of asset correlations

(horizontal axis: the number of companies)

— Original values of asset correlation are 0.01, 0.10 and 0.20 on the upper, middle and lower row, respectively.

Asymptotic Moment Method | Finite Moment Method | Maximum Likelihood Estimation



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[36] When constructing Figure 16, we used the Gaussian Quadrature (150 divisions) and formulas (13)–(15).

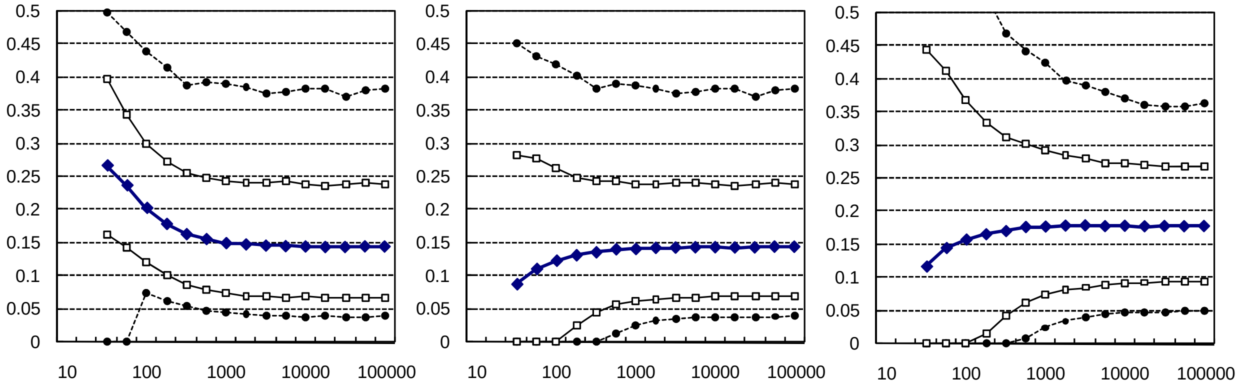

Figure 16 shows the following results.

(1) When the number of companies is small, the Asymptotic Method of Moments tends to estimate larger asset correlations than the original asset correlation. When the default rate is set at 1% and the number of companies is more than about 1,000, all the methods tend to obtain almost the same result for the reestimated asset correlation.

(2) When the number of companies is very large, using any method the estimated value of asset correlation is below its actual value, especially when setting large asset correlations.

(3) When the number of companies is very large, the shapes of the functions with the Finite Method of Moments and Maximum Likelihood Estimation are similar, but the average value of the results using Maximum Likelihood Estimation is closer to the original value than with the Finite Method of Moments. In addition, the variance of results with Maximum Likelihood Estimation is smaller than with the Finite Method of Moments.

In terms of (1), when we calculate the variance with the default rate calculated by $\overline{n}_k$ number of companies, the Finite Method of Moments $\mathrm{E}_X[p_k(x)(1-p_k(x))/\overline{n}_k]$ differs from Maximum Likelihood Estimation (see formula (11)). In particular, when the number of companies is small, the estimated asset correlation tends to be too large. Our result that when $\overline{n}_k$ is larger than 1,000, the estimated result is almost the same as the actual value is adjusted to the result of Düllmann and Scheule [2003].

In terms of (2), Gordy and Heitfield [2002] and Demey, Jouanin and Roget [2004] obtain the result that when the number of companies is large, the estimated value is slightly below the actual value[37].

Next, we analyze the relation between term t and the estimated value. We set the asset correlation at 0.1 and the default rate at 0.01, and t is 10, 15, 20 and 30 years; we then calculate the asset correlation using the Asymptotic Method of Moments, Finite Method of Moments and Maximum Likelihood Estimation. The number of companies takes three patterns, 1,000, 10,000 and 100,000. Table 7 shows the results.

[Table 7] Differences in results with differences in the number of companies and the data term

— We set the asset correlation at 0.1 and the other parameters as in Table 6.

Asymptotic Method of Moments

	10 yr	15 yr	20 yr	30 yr
1,000 comp.	0.0907	0.0947	0.0984	0.1006
10,000 comp.	0.0838	0.0879	0.0901	0.0930
100,000 comp.	0.0833	0.0870	0.0894	0.0930

Finite Method of Moments

	10 yr	15 yr	20 yr	30 yr
1,000 comp.	0.0808	0.0852	0.0893	0.0917
10,000 comp.	0.0828	0.0869	0.0891	0.0921
100,000 comp.	0.0832	0.0869	0.0893	0.0929

Maximum Likelihood Estimation

	10 yr	15 yr	20 yr	30 yr
1,000 comp.	0.0891	0.0928	0.0950	0.0968
10,000 comp.	0.0898	0.0933	0.0944	0.0963
100,000 comp.	0.0898	0.0926	0.0945	0.0964

Table 7 shows that when the data term is short, the estimated asset correlation is too small. The longer the data term, the more precisely the correlation is estimated. Demey, Jouanin and Roget [2004] also obtain the result that the larger t is, the closer the estimated value of asset correlation is to the original value.

In terms of (3), we can understand the result of (3) using Figure 16. In addition, Table 8 shows that the variance of distribution of estimated asset correlation is smallest using Maximum Likelihood Estimation among these three methods.

[Table 8] Variance of distribution of estimated asset correlation

— We set the asset correlation at 0.1 and the other parameters as in Table 6.

Original asset correlation	Asymptotic Method of Moments	Finite Method of Moments	Maximum Likelihood Estimation
0.2	0.0722	0.0722	0.0682
0.1	0.0419	0.0420	0.0381
0.01	0.0048	0.0048	0.0043

[37] We have tried to no avail to find why the estimated value is below the original value when there are many companies.

Appendix 4. The number of data for estimation of asset correlation

This appendix shows the number of data (mean, from 1985 to 2005) in the Teikoku Data Bank's Matrix Data which we use for estimating asset correlations.

(1) The number of data in each industry

[Table 9] The average number of companies in each industry

— Above: the number of companies, Below: the number of default companies

All Industry	Agriculture, forestry, hunting, fishery and mining	Construction	Manufacturing	Retail wholesale, restaurant
960,980	9,515	194,377	175,173	382,660
7,677	59	2,170	1,348	2,853

Finance	Real estate	Transportation, telecommunications, electricity, gas, etc.	Service
1,160	41,960	34,318	121,817
11	270	248	718

(2) The number of data by company size

[Table 10] The average number of companies by each company size

— Above: the number of companies, Below: the number of default companies

Large and medium-sized	Small	Personal
22,384	792,900	145,697
74	6,843	759

[Table 11] The average number of companies by each size in the main industries

— Above: the number of companies, Below: the number of default companies

	Large and medium-sized	Small	Personal
Manufacturing			
	2,589	157,260	15,324
	2	1,229	117
Construction			
	446	166,599	27,333
	2	2,011	157
Retail, wholesale and restaurant			
	9,482	293,137	80,041
	36	2,407	410
Service			
	9,254	96,547	16,016
	33	641	44

(3) The number of data in Score on a scale with increments of five

[Table 12] The number of companies by Score
— Above: the number of companies, Below: the number of default companies

100-96	95-91	90-86	85-81	80-76	75-71	70-66	65-61	60-56	55-51
1	13	86	276	1,334	5,674	16,031	39,936	112,844	321,609
0	0	0	0	0	1	4	28	200	1,431
50-46	45-41	40-36	35-31	30-26	25-21	20-16	15-11	10-6	5-0
227,151	137,333	65,689	20,591	5,585	1,021	196	29	6	5,577
2,112	1,894	1,254	490	171	36	7	1	0	47

(4) The number of data in Score on a scale with increments of five

[Table 13] The number of companies in each region
— Above: the number of companies, Below: the number of default companies

Japan (as a whole)	Hokkaido / Tohoku	Kanto–Koshinetsu	Hokuriku	Chubu	Kinki	Chugoku	Shikoku	Kyushu/ Okinawa
960,980	112,308	28,655	381,471	100,924	150,549	62,410	31,914	92,748
7,677	964	184	3,067	637	1,354	434	202	835

[Table 14] The number of companies in each prefecture
— Above: the number of companies, Below: the number of default companies

Hokkaido	Aomori	Iwate	Miyagi	Akita	Yamagata	Fukushima	Ibaraki
47,609	10,537	8,854	15,049	7,953	8,007	14,298	18,237
447	82	59	136	76	59	105	96
Tochigi	Gunma	Saitama	Chiba	Tokyo	Kanagawa	Niigata	Toyama
14,351	14,342	39,073	27,986	173,982	48,529	20,159	9,406
90	89	305	217	1,589	385	137	60
Ishikawa	Fukui	Yamanashi	Nagano	Gifu	Shizuoka	Aichi	Mie
9,754	9,494	7,865	16,947	13,536	26,999	48,455	11,934
68	56	56	104	84	161	323	69
Shiga	Kyoto	Osaka	Hyogo	Nara	Wakayama	Tottori	Shimane
6,620	19,558	78,828	32,189	5,919	7,436	4,705	5,329
40	160	805	255	47	48	36	32
Okayama	Hiroshima	Yamaguchi	Tokushima	Kagawa	Ehime	Kochi	Fukuoka
16,483	24,386	11,507	6,501	8,434	10,963	6,016	33,756
101	189	75	39	50	71	41	310
Saga	Nagasaki	Kumamoto	Oita	Miyazaki	Kagoshima	Okinawa	
5,546	9,606	11,033	8,725	7,942	9,968	6,172	
47	80	94	72	73	88	71	

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