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A Global Game Analysis of Emergent Liquidity Provision and the Role of Creditors' Aggregate Behavior as Signaling

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Abstract

Recent funding problems experienced by European sovereigns and the subsequent policy actions have renewed interest in emergent liquidity provision or the international Lender of Last Resort. This paper constructs an abstract model about emergent liquidity lending by using global game techniques. Compared with the existing models, our model can be characterized by the followings: (1) the authority to provide liquidity (policy maker) is an explicit player in the game rather than an implicit unity appeared in comparative statics, (2) the policy maker cannot distinguish between solvency and insolvency of the liquidity borrower ex ante, (3) liquidity lending rates are endogenously determined, and (4) the policy maker's decision making is set after observing creditors' aggregate behaviors of withdrawing their loans to the borrower. With this setup, it is shown that: (1) creditors' aggregate behavior operates as a signal to the policy maker about borrower's solvency, (2) the policy maker's optimal strategy is to help only illiquid but solvent borrowers, (3) whenever the liquidity lending facility is utilized, optimal lending rates are strictly positive, and (4) the optimal lending rates are "conditionally punitive" in the sense that they take the highest level possible under the restriction that the rates enable solvent but illiquid borrowers to survive.

JEL classifications: C72, G28

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1 Introduction

Recent economic and financial situations have greatly renewed interest in the role of emergent liquidity provision in both international and domestic contexts. On the international side, the emergence of the fiscal deficit problem in Greece triggered global concern about sovereign risks [1]. The fiscal problem in Greece spread to other peripheral European countries where the fiscal conditions were also deteriorating, and this inevitably drew great attention to the feasibility of future funding by European sovereigns and effective framework of emergent liquidity lending or international Lender of Last Resort (LLR), which has been examined as an international crisis manager in such works as Kindleberger [23] and Fischer [14]. Turning to the domestic context, a bank run on the *Northern Rock* in the United Kingdom and the New York Fed's emergent lending to *Bear Stearns* through *JPMorgan* in the United States are typical examples that have provoked lively discussions on the role of the domestic emergent liquidity lending. In this paper, we will construct an abstract model about emergent liquidity lending by using global game techniques.

In empirical works, the evidence on the emergent liquidity lending points unambiguously to the conclusion that it has helped to avoid global financial crises or bank panics. Miron [24] investigates the effects of the founding of the Federal Reserve Bank (Fed) in the United States on bank panics and shows that, after it was founded in 1914, the frequency of financial panics declined substantially. Friedman [16] pointed out that during the period of 1929-1933, money stock shrank by 31 percent and the price level declined by 25 percent primarily because the Fed failed to act as the lender of last resort. Bordo [7] examines the changes that occurred in the United States and the United Kingdom before and after the creation of an LLR function. In the United Kingdom case, for instance, he shows that the Bank of England's action as an LLR prevented rudimentary financial crises in 1878, 1890 and 1914 from becoming more severe panics. Eichengreen and Portes [13], among others, also support this view.

After Thornton [31], who was the first advocate of the LLR and identified the Bank of England's characteristics as an LLR, the theory of the LLR received its most influential

[1] As regards recent developments of sovereign risks in European and other countries, see Bank of Japan [6] and Shino and Takahashi [30].

exposition in Bagehot's statement [3] during the nineteenth century. The essentials of the concept can be succinctly summarized as follows.

- The LLR should aim to prevent illiquid but solvent banks from failing.
- Lending should be provided without limit but at a penalty rate ("very large loans at very high rates are the best remedy").
- Lending must be open to any solvent banks provided that they have good collateral.
- It should be made clear in advance that the LLR authority is ready to lend freely.

While this "classical view" on the LLR has been widely supported by many empirical works, Bagehot's argument has also been subject to some criticisms. Essentially, according to Bordo [7] and Freixas and Rochet [15], all of the criticisms can be divided into the following categories.

- Goodhart [18] [19] [20] argues that it is impossible for an LLR authority to distinguish solvent from insolvent banks and, taking the contagious nature of financial panics into account, any banks, including insolvent ones, should be rescued.
- Goodfriend and King [17] argue that with a well-developed money market, open market operations are sufficient in providing liquidity to solvent banks and, thus, LLR is not needed.
- Proponents of free banking argue that legal restrictions (prohibition of free currency issues, for example) are the only source of banking panics, and emphasize the need to establish free market mechanisms, rather any active interventions by government, including the LLR.

For those criticisms, the pioneering work of Rochet and Vives [28], hereafter R&V, revisits and revives Bagehot's assertion. Using a modern methodological framework of a "global game [2]," R&V establish a theoretical foundation regarding the need for the LLR and provide

[2] A global game is one of the pioneering fields for incomplete information games. Other applications include currency crises (Morris and Shin [25]), debt crises (Corsetti, Guimaraes, and Roubini [10]), investment (Chamley [9]; Dasgupta [11]), liquidity crashes (Morris and Shin [27]), and sociopolitical change (Atkeson [2]; Edmond [12]). See also Carlsson and van Damme [8] for their pioneering contribution on global games and Morris and Shin [26] for recent developments.

a rationale for Bagehot's doctrine of helping illiquid but solvent banks.

Specifically, their model is as follows [3]. There is a market with three dates in which two types of players exist: (a continuum of) depositors/creditors and a liquidity borrower (typically commercial bank, hereafter PB). At the first stage, the return on the PB's investments or loans on the assets side of its balance sheet, which is assumed to be normally distributed, is realized. The returns on these assets are collected at the end of the game. At the second stage, observing the realization of the return, each agent decides to withdraw his or her deposit. If the amount of cash needed for the repayment at this interim stage exceeds the amount of cash that the PB initially has, the bank has to sell some of its (noncash) assets in an asset market. Therefore, even if the return is high enough for the bank to be solvent, it could experience a liquidity shortage. If the PB cannot obtain enough cash even when all its assets are sold, it fails at the second stage. At the last stage, all remaining depositors are assumed to withdraw their deposits. If the amount of cash needed for the repayment exceeds the sum of cash (if the bank has any) and the return collected, then the PB fails.

If each agent can observe the realization of the return perfectly (without noise) before their decision making, multiple equilibria could exist: for a certain range of realizations of the return, both outcomes of "all agents withdraw" and "all agents do not withdraw" are equilibria. To avoid such a multiplicity or coordination problem, the global game elaborates to introduce a *small noise* associated with the depositors' observation. Following this procedure, R&V show that there exists a unique Bayesian equilibrium where the probability that a solvent bank fails due to a liquidity shortage is strictly positive. They further implement a comparative statics analysis for the unique equilibrium and show that emergent liquidity provision (or LLR policy) effectively works in decreasing the probability of failure of such solvent banks experiencing liquidity shortages to arbitrarily close to zero. In other words, they succeed in rejuvenating Bagehot's doctrine of helping illiquid but solvent banks.

Our paper builds on their model, but it diverges in several key aspects. Furthermore, our results support not only the above statement about solvency but also Bagehot's other

[3] While their model is applicable to both international and domestic contexts, for simplicity we explain the model in line with the domestic context. The model will be reinterpreted in an international setting later in Section 2.

assertions, which are also valid in terms of historical records of emergent liquidity provision taken by international institutions or central banks. The main points are as follows.

First, in R&V, the policy maker's behavior is treated in terms of comparative statics and it is not taken as an explicit player engaged in interactions between PBs or depositors. In the existing literature on global games, however, the importance of endogenizing the policy maker's action is emphasized, as seen in Angeletos et al. [1]. In that model, a policy maker's action is taken before the agents' interaction thus the policy choice could work as a signal to agents. In contrast to this, we construct a model in which agents move first and, then, after observing their aggregate behaviors, the policy maker takes action. Such a construction is especially appropriate for analyzing emergent liquidity provision or LLR policy because the policy maker usually provides an LLR scheme after depositors/creditors behavior of withdrawing their funds. Here, it is worth noting that while the order of the play is the same as the currency attack model of Morris and Shin [25], our model crucially differs in that the policy maker cannot observe the true realization of the return (the fundamentals). Therefore, in our model, agents' aggregate action could work as a signal to the policy maker about the fundamentals. Specifically, we will examine the situation where the policy maker has no information about the return but can observe the *exact* proportion of depositors who withdraw. In such a case, we will see that the depositors' aggregate behavior works perfectly as a signal so that the policy maker can predict the true realization of returns.

Next, by introducing the policy maker as an explicit player, two related issues emerge: the strategy set and the utility function of the policy maker. In regard to the strategy set, we assume that not only is there a choice regarding whether to provide the LLR facility, but the lending rate is also a choice variable of the policy maker. This contrasts with, for example, Corsetti et al. [10], in which the policy maker's choice is assumed to be the binary one — provide the LLR or not — with a fixed rate. Our model in which the lending rate is endogenously determined as the policy maker's optimal behavior is especially powerful in examining the validity of Bagehot's statement regarding "penalty rate" lending.

As for the policy maker's utility function, we assume that it has two components: the *financial stability term* and the *balance sheet soundness term*. The former is simply to specify that the policy maker prefers the PB's survival to its failure. The latter presents the costs or benefits in terms of financial soundness of the policy maker's balance sheet. For example, if the policy maker lends liquidity to PB and it eventually fails, the policy maker incurs a financial cost because the fund would not be repaid.

With this setup, our main results support Bagehot's classical statements and provides some observations on how the facility works. First, we show that the policy maker's optimal behavior is to help only illiquid but solvent borrowers. While R&V assume that the policy maker, which is treated as an implicit unity in the comparative statics, can distinguish solvent from insolvent borrowers ex ante, our model clarifies the mechanism through which the signaling role of depositors' behavior enables the policy maker to distinguish these and to implement the optimal policy.

Second, it is shown that the optimal lending rates are always strictly positive whenever the lending facility is utilized by PBs. Furthermore, the optimal lending rates can be seen as "conditionally punitive" ones in the sense that they take the highest level possible under the restriction that the rates enable solvent but illiquid PBs to survive. Such a punitive rate is attained via the policy maker's balance sheet soundness term, while the restriction comes from the financial stability term, both of which are embedded in its utility function. These results provide new insights and seem to be consistent with actual policy operations of emergent liquidity provision and Bagehot's assertion to "*lend, but at a penalty rate*". It may also be worth mentioning that Humphrey [22] points out several advantages of lending at relatively high (not zero) rates in terms of borrower's moral hazard.

To this end, in this paper, we construct a global game model so that these issues are appropriately taken into consideration, and derive an equilibrium that supports Bagehot's classical statements and the actual operations of emergent liquidity provision. The organization of the rest of the paper is as follows. In Section 2, we formally explain the model. Section 3 shows the main results. Some concluding remarks are made in Section 4.

2 The Model

This section formally constructs an abstract model about emergent liquidity provision by using global game techniques. While the model is abstract and applicable to both domestic and international contexts, for simplicity we explain the model in line with the domestic context. The model will be reinterpreted in an international setting later in this section.

Our model builds on R&V, but differs in several aspects that can be characterized by the following assumptions: (1) the authority to provide liquidity (policy maker) is an explicit player whose preference is based on the soundness of its own balance sheet as well as whether the PB fails, (2) the authority cannot distinguish the solvency and insolvency of the PB ex ante, (3) the lending rates are endogenously determined, and (4) the policy maker's decision making is set after observing depositors' aggregate behaviors.

Consider a market with five dates ($\tau = 0, 1, 2, 3, 4$). There are three types of players. The first one is the policy maker to provide liquidity or LLR scheme. We occasionally denote the policy maker by *PM*. Other types are a commercial bank (hereafter *PB*) and a continuum of *depositors* (*agents*) of measure one, indexed by i and uniformly distributed over $[0, 1]$. Initially, the PB possesses its own funds E and collects uninsured deposits for some amount D_0 that is normalized to one. These funds are used in part to finance some investment I in risky loans, the rest being held in cash reserves M. $M < D_0$ is assumed throughout this paper and the PB's balance sheet is represented as follows.

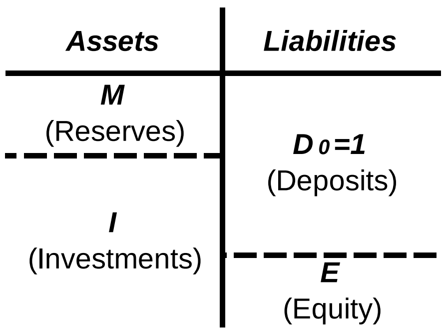

Figure 1: The PB's Initial Balance Sheet

The nominal value of deposits on withdrawal is $D \geq D_0$, independent of the withdrawal date. Thus, early withdrawal entails no cost for the depositors themselves. For simplicity, $\frac{M}{D} < \frac{1}{2}$ is assumed [4].

In the last period ($\tau = 4$), the returns on the investment RI are collected, the deposits are repaid and the stockholders of the bank receive the difference when this is possible. However, early withdrawals may occur at $\tau = 1$, following private observations on the realization of R. Specifically, the precise timeline of the game is described as follows.

<u>$\tau = 0$</u> $R \sim N(\bar{R}, 1/\alpha)$ (a normal distribution with the mean $\bar{R}$ and the variance $1/\alpha$) is realized. The distribution is the common prior belief among the PM and all agents.

<u>$\tau = 1$</u> Observing $s_i = R + \epsilon_i$, where $\epsilon_i \sim N(0, 1/\beta)$, agent i decides whether to withdraw his or her deposit (W) or not (NW). Thus, i's strategy is defined as $A_i : S_i \longrightarrow \{W, NW\}$, where S_i is defined as all possible observations. Let x be the proportion of agents that choose W.

<u>$\tau = 2$</u> Observing x and with no information about R[5], the PM decides whether to provide an LLR scheme of lending money at a rate r[6] without limit (r), or not to provide the scheme (NL). Thus, the PM's strategy is defined as $A_{PM} : X \longrightarrow \{NL \cup \Re^+\}$.

<u>$\tau = 3$</u>

(1) If $xD \leq M$, the PB has enough cash for repayments at $\tau = 1$ and, thus, the PB only uses the amount of xD of its own cash. In this case, the PB never fails at $\tau = 3$.

(2) Suppose $xD > M$. In the case of such a liquidity shortage, the PB may sell some of its assets in a secondary market. Following R&V, this secondary market for bank assets is assumed to be informationally efficient in the sense that the secondary price aggregates the decentralized information of investors about the quality of the PB's assets. This means that the resale value of the bank's assets depends on R. However, the bank cannot obtain the full value of its assets, but only a fraction $\frac{1}{1+\lambda}$ of this value, with $\lambda > 0$.

[4] This does not violate the cash–deposit ratios in actual banking operations.

[5] Here, we impose the extreme assumption that the PM observes no information about R. However, as we shall show, as long as the PM can observe the true x, the PM can predict the true realization of R through x. In other words, a depositors' aggregate behavior expressed as x works perfectly as a signal about R. Thus, no matter which assumption is imposed on the PM's observation about R, all of the results derived later hold.

[6] In Corsetti [10], the lending rate is fixed and normalized to zero. By assuming the lending rate is a *choice variable* of the PM, we can analyze whether a positive (penalty) rate can be justified as the PM's optimal behavior.

Accordingly, the volume of sales needed in the face of withdrawals x is given by:

$$y(x) \equiv (1+\lambda)\frac{xD - M}{R}. \tag{1}$$

Thus, all possible cases depending on the PM's actions are described as follows.

(a) If the LLR policy is not provided at $\tau = 2$, the PB sells some of its assets $y(x)$. The PB fails at this period when $y(x) > I$.

(b) Suppose that the LLR policy is provided. In this case, failure never occurs at $\tau = 3$. Furthermore, the following apply.

   i. If the policy is costless[7], rather than selling its assets, the PB borrows $xD - M$ at the rate r.

   ii. If the policy is more costly than selling its assets and $y(x) \leq I$, the PB only sells $y(x)$.

   iii. If the policy is more costly than selling its assets and $y(x) > I$, the PB sells all its assets and borrows $\left(xD - M - \frac{IR}{1+\lambda}\right)$ from the PM at the rate r.

<u>$\tau = 4$</u> For each case, in the previous period, the PB fails when the following apply.

(1) $RI + (M - xD) < (1-x)D$.

(2)-(a) $y(x) \leq I$ and $R(I - y(x)) < (1-x)D$.

(2)-(b) i. $RI - (1+r)(xD - M) < (1-x)D$.

   ii. $R(I - y(x)) < (1-x)D$.

   iii. Always.

Finally, we need to specify an agent's and the PM's payoff structures. The former is identical to R&V, as depicted in Fig.2.

[7] Suppose the PM provides the LLR scheme and $xD > M$. Then, the PB faces a liquidity shortage and has two choices: borrowing from the PM at the rate r, or selling $y(x)$ of its own assets. We say that the LLR rate is "costless" for the PB if borrowing from the PM is strictly profitable in terms of the PB's total assets at the final period, that is,

$$RI - (1+r)(xD - M) > R(I - y(x)) \Leftrightarrow r < \lambda.$$

Thus, the PB uses the LLR scheme only when the lending rate is less than the liquidity premium.

	Failure	No Failure
W	B_a	$-C_a$
NW	0	0

Figure 2: An Agent's Payoff

Here, $B_a > 0$ and $C_a > 0$. This means that when the PB fails, the (normalized) payoff from withdrawing is B_a, whereas the payoff from withdrawing when the bank does not fail is $-C_a$. The underlying situation behind the setup is that an agent's payoff depends on whether he or she makes the "right decision" (see [28]).

In contrast to R&V, we need to specify the PM's payoff structure because the PM is taken as an explicit player in the game. First, note that given the LLR scheme provided, the amount of lending money depends on x (and parameters). We denote the money by $b(x)$.

Now, we assume that the payoff structure of the PM consists of two terms: the *financial stability term* and the *PM's balance sheet soundness term*. The financial stability term represents the fact that the PM prefers the PB's survival to its failure, and is depicted in Fig.3.

	Failure	No Failure
L	0	B_{PM}
NL	0	B_{PM}

Figure 3: Financial Stability Term

Here, $B_{PM} > 0$. This means that when the PB survives, the PM gets benefit B_{PM} whereas the loss of failure is normalized to zero.

The PM's balance sheet soundness term, on the other hand, is defined as in Fig.4.

	Failure	No Failure
L	$-b(x)$	$rb(x)$
NL	0	0

Figure 4: PM's Balance Sheet Soundness Term

If the PB borrows $b(x)$ at $\tau = 3$ and fails at $\tau = 4$, the loss of the PM's balance sheet is $b(x)$. On the other hand, if the PM lends $b(x)$ with the rate r and the money is repaid, the "soundness" of the PM's balance sheet improves by the amount of $rb(x)$. This specification builds on the view that conducts of emergent liquidity provision or monetary policy which takes the balance sheet condition of the policy maker (typically central bank) into account will ultimately benefit the taxpayers. This view have continuously emphasized by policy makers at international institutions and central banks around the world. One of the latest examples can be seen in Sack [29]. Another example is the Bank of Japan's announcement [5] about the basic principles of its LLR policies [8].

Finally, we assume that these two terms are additive and, thus, the PM's payoff structure is given as in Fig.5

	Failure	No Failure
L	$-b(x)$	$B_{PM} + rb(x)$
NL	0	B_{PM}

Figure 5: The PM's Payoff Structure

Note that such a construction of the PM's payoff function is possible because the lending rate is a choice variable of the PM. This contrasts with, for example, Corsetti et al. [10], in which the policy maker's choice is binary. A variable lending rate is an appropriate setting, especially in examining the validity of Bagehot's statement of the "penalty rate" for lending.

Finally, we reinterpret the model in an international setting and provide a potential rationale for an international LLR a la Bagehot. Suppose now that the balance sheet of Fig.1 corresponds to a small open economy where D_0 is the foreign denominated short-term debt, M is the amount of foreign reserves, I is the investment in risky local entrepreneurial projects, E is equity and long-term debt (or local resources available for investment) and D is the face value of the foreign denominated short-term debt. The depositors in the domestic setting are now international fund managers operating in the international interbank market. The liquidity ratio $m = \frac{M}{D}$ is now the ratio of foreign reserves to foreign short-term debt, which is a crucial ratio, according to empirical work, in determining the probability of a crisis in a country. The parameter λ now represents the fire sales premium associated with early sales of domestic bank assets in the secondary market. With the international setting, the financial

[8]"Four Principles in Conducting Business Necessary to Maintain Financial Stability "of Bank of Japan [5] states that "the financial soundness of the Bank of Japan should not be impaired."

stability term depicted in Fig.3 means that an international LLR authority such as the IMF prefers the small economy's survival to its failure.

3 Main Results

3.1 The PB's States on the (R, x) Plane

For each PM's choice at $\tau = 2$, we can plot different PBs' states on the (R, x) plane. To do so, it is convenient to divide all of the PMs' actions into three cases: (1) not providing the LLR scheme, (2) providing the scheme at a low rate ($r < \lambda$), and (3) providing the scheme at a high rate ($r \geq \lambda$).

(1) If the PM does not provide the LLR scheme, our results are no different from those of R&V. Lines (A) and (B) in Fig.6 [9] are given by the following:

$$
\begin{aligned}
Line(A): \; R &= (1+\lambda)\left(\frac{xD - M}{I}\right) \;\; (\Leftrightarrow y(x) = I) && (2) \\
Line(B): \; R &= R_s + \lambda\left(\frac{xD - M}{I}\right), && (3)
\end{aligned}
$$

where $R_s \equiv \frac{D-M}{I}$.

In the case of (2) and (3), if $xD \leq M$, the shape of the figures is identical to (1) because the PB has enough cash for early withdrawals (the PB never uses the scheme even when it is available). Now, suppose that $xD > M$ (the liquidity shortage case). First, if the LLR lending rate is low, failure at $\tau = 3$ never occurs and the PB fails at $\tau = 4$ if and only if $RI - (1 + r)(xD - M) < (1 - x)D$. By arrangement, we get the following Line (C) in Fig.7:

$$
Line(C): \; R = R_s + r\left(\frac{xD - M}{I}\right). \qquad (4)
$$

Note that failure occurs when R is strictly less than the right-hand side (RHS) of (4) and Lines (B) and (C) differ only in slopes. Next, suppose that $xD > M$ and the LLR rate is high. In this case, if $y(x) \leq I$, then the PB does not use the LLR scheme. Thus, the shape of the chart is

[9] Fig.6 is identical to Fig.2 in Rochet and Vives [28].

identical to (1) (right to Line (A)). On the other hand, if $y(x) > I$, then the PB uses the scheme after selling all its assets and it always fails at $\tau = 4$.

Here, it is worth explaining the notions of "solvency" and "illiquidity" in our model. Suppose that no early withdrawal occurs at $\tau = 1$, so that $x = 0$. In our setup, the PB is solvent if and only if $R \geq R_s$. Therefore, R_s can be regarded as the minimum rate of return at which the PB is solvent, conditional on no liquidity drain occurring in the interim period. Furthermore, suppose that $R \geq R_s$ (and x is not necessarily zero). If, in Fig.6, x is located above Line (B) ($R < R_s + \lambda\left(\frac{xD-M}{I}\right)$), the PB would fail because of illiquidity although its solvency is retained. We will show that such a solvent but illiquid PB is always rescued with the LLR at a strictly positive lending rate in every equilibrium.

Finally, for the following analysis, let $\hat{x}$ be the solution for $R_s = (1+\lambda)\left(\frac{xD-M}{I}\right)$, that is:

$$\hat{x} = \frac{D + \lambda M}{(1+\lambda)D}. \tag{5}$$

Note that $\hat{x} > \frac{M}{D}$.

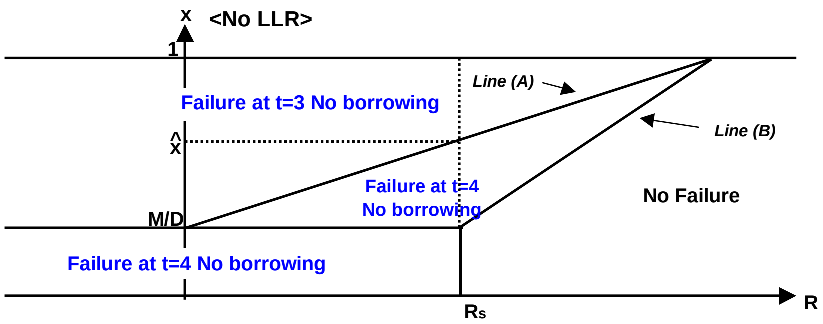

Figure 6: States of the PB on the (R, x) Plane (No LLR Scheme)

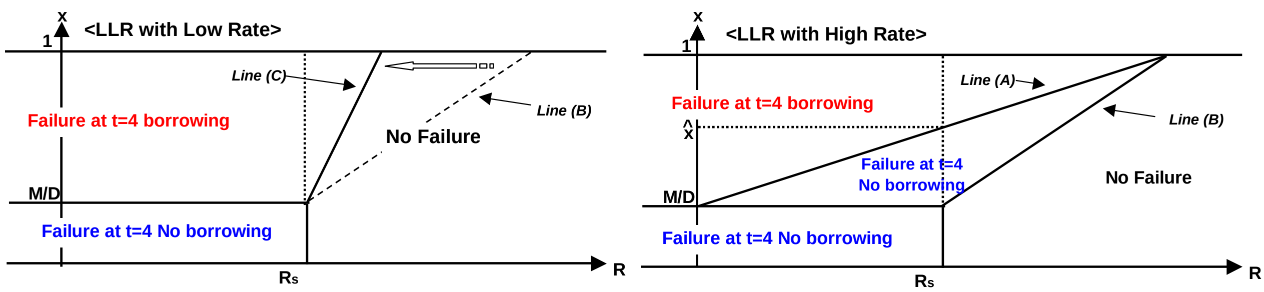

Figure 7: States of the PB on the (R, x) Plane (Left: LLR with a Low Rate; Right: LLR with a High Rate)

3.2 Agents' Strategy and Aggregate Behavior

Now, we derive a symmetric perfect Bayesian equilibria. Following the basic procedure in global game analysis, we focus on an equilibrium in which each agent uses a *threshold strategy*:

$$A_i(s_i) = \begin{cases} W & \text{if } s_i < t \\ NW & \text{if } s_i \geq t, \end{cases} \tag{6}$$

which means that agent i withdraws if and only if his or her signal is below some threshold t. Suppose that all agents use the same threshold strategy with t, denoted by $A(s_i)$. Then, the probability of agent i choosing W conditional on R is given by:

$$P(withdraw \mid R) = P(s_i < t|R) = P(\epsilon_i < t - R|R) = G_\epsilon(t - R), \tag{7}$$

where G_ϵ is *c.d.f.* of $\epsilon \sim N(0, 1/\beta)$. From the law of large numbers, this probability is identical to the proportion of agents that withdraw under t conditional on R. Furthermore, because G_ϵ is a strictly increasing function and thus one to one correspondence, we can define its inverse function G_ϵ^{-1}. As a result, given t, the PM can compute the true R by:

$$R(x, t) = t - G_\epsilon^{-1}(x). \tag{8}$$

Note that $R(x, t)$ is strictly decreasing in x and increasing in t. Figs.8 and 9 exhibit visual images of $G_\epsilon(\cdot)$, $G_\epsilon^{-1}(x)$, $-G_\epsilon^{-1}(x)$ and $R(x, \tilde{t}) = \tilde{t} - G_\epsilon^{-1}(x)$ for a given $\tilde{t}$ and two different β, where $\underline{\beta} < \overline{\beta}$.

Now, we provide a brief interpretation of $R(x, t)$ by using the right-hand graph of Fig.9. First, suppose that the PM observes $\tilde{x}$ with $\tilde{x} < \frac{1}{2}$. As less than half of the agents withdraw their deposits, the PM predicts, assuming that $t = \tilde{t}$, that more than half the signals are greater than the threshold $\tilde{t}$. Therefore, R is predicted to be greater than $\tilde{t}$. Thus, $R(x, \tilde{t}) > \tilde{t}$ when $\tilde{x} < \frac{1}{2}$. Similarly, $R(x, \tilde{t}) < \tilde{t}$ when $\tilde{x} > \frac{1}{2}$. Note that this observation is true, independent of β.

Next, consider the effects of different values of β $\left(\underline{\beta} < \overline{\beta}\right)$ on $R(x, \tilde{t})$. For a given $\tilde{R}$ with $\tilde{R} < \tilde{t}$, the proportion of agents who do not withdraw their deposits when $\beta = \underline{\beta}$ is greater than when $\beta = \overline{\beta}$ (see Fig.10). Therefore, denoting $R_\beta(x,t) \equiv R(x,t;\beta)$, this means that the line $R_{\underline{\beta}}(x,\tilde{t})$ is located to the left of $R_{\overline{\beta}}(x,\tilde{t})$ when $x > \frac{1}{2}$. Similarly, when $x < \frac{1}{2}$, the line $R_{\underline{\beta}}(x,\tilde{t})$ locates to the right of $R_{\overline{\beta}}(x,\tilde{t})$ (see Fig.9).

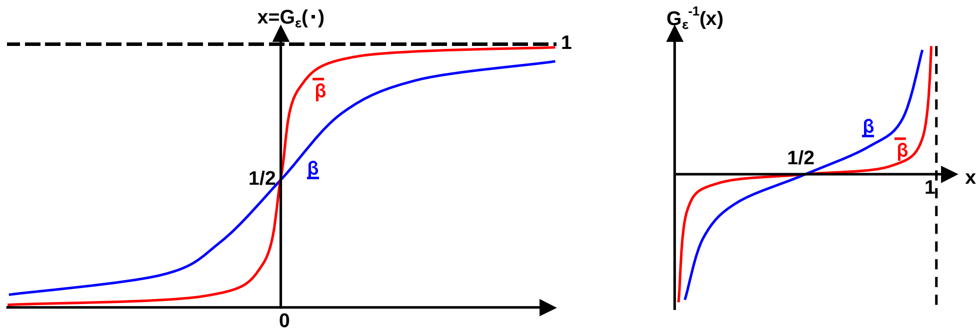

Figure 8: Shapes of $x = G_\epsilon(\cdot)$ (left) and $G_\epsilon^{-1}(x)$ (right)

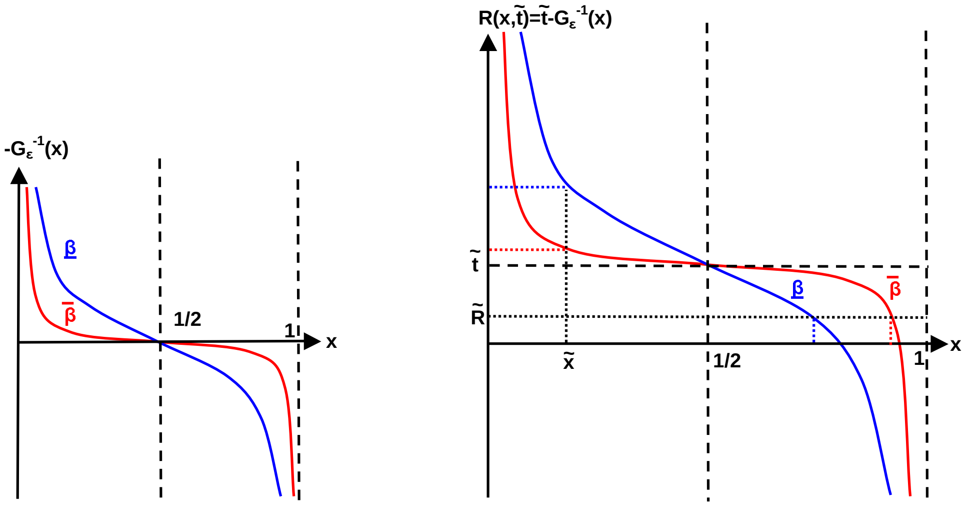

Figure 9: Shapes of $-G_\epsilon^{-1}(x)$ (left) and $R(x,t) = t - G_\epsilon^{-1}(x)$ (right)

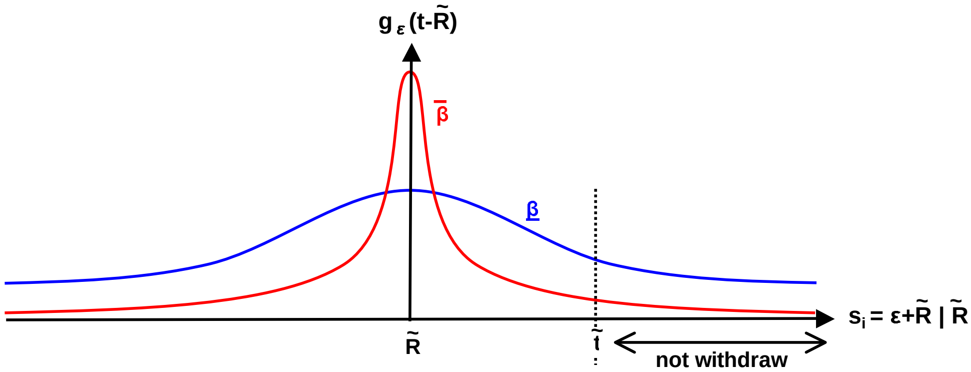

Figure 10: The Proportion of Agents that do (not) Withdraw for Different β

It might be worth noting here that the PM needs to know which t is adopted by the agents in order to infer the true R. We will see in Section 3.5 that the PM can actually compute the optimal t, depending on the parameters.

3.3 The PM's Best Responses

For a given t, the PM can compute the best response at every observation of x. As an example, see Fig.11 and suppose that the agents' threshold is $\tilde{t}$ and that the PM observes $\tilde{x}$. Then, the PM can correctly predict that $\tilde{R}$ by $\tilde{R} = R(\tilde{x}, \tilde{t}) = \tilde{t} - G_\epsilon^{-1}(\tilde{x})$. If the PM does not provide an LLR regime or does so with a "high rate", the PB fails and, thus, the PM obtains 0 (no LLR) or $-b(\tilde{x})$ (LLR with a high rate). On the other hand, if the PM proposes an LLR regime with a low rate, so that the intersection of *Line (C)* and $R(x, \tilde{t})$ is to the left of $\tilde{R}$ (see the bottom of Fig.11), then the PB survives by using the LLR scheme and, thus, the PM gets the positive benefit of $B_{PM} + rb(\tilde{x})$. Furthermore, from the PM's balance sheet soundness term, it is obvious that placing $Line(C^*)$ such that $(\tilde{R}, \tilde{x})$ lies on it is PM's unique best response for a given $\tilde{t}$ and $\tilde{x}$ (recall that (4): the "No failure" region includes $Line(C^*)$ and the "failure at $\tau = 4$ borrowing" region does not). Note that, in this case, the lending rate is strictly positive and the "solvent but illiquid" bank is rescued. Now, we formally derive the PM's best response. First, we divide all ts into the following four cases (see also Fig.12).

Case (a)	:	$R\left(\frac{M}{D}, t\right) < 0$	**Case (c)**	:	$R\left(\frac{M}{D}, t\right) \geq R_s$	and	$R(\hat{x}, t) < R_s$
Case (b)	:	$0 \leq R\left(\frac{M}{D}, t\right) < R_s$	**Case (d)**	:	$R\left(\frac{M}{D}, t\right) \geq R_s$	and	$R(\hat{x}, t) \geq R_s$.

Each case is identical to: (a) $t < G_\epsilon^{-1}\left(\frac{M}{D}\right)$, (b) $G_\epsilon^{-1}\left(\frac{M}{D}\right) \leq t < G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s$, (c) $G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s \leq t < G_\epsilon^{-1}(\hat{x}) + R_s$ and (d) $G_\epsilon^{-1}(\hat{x}) + R_s \leq t$ (see Fig.13).

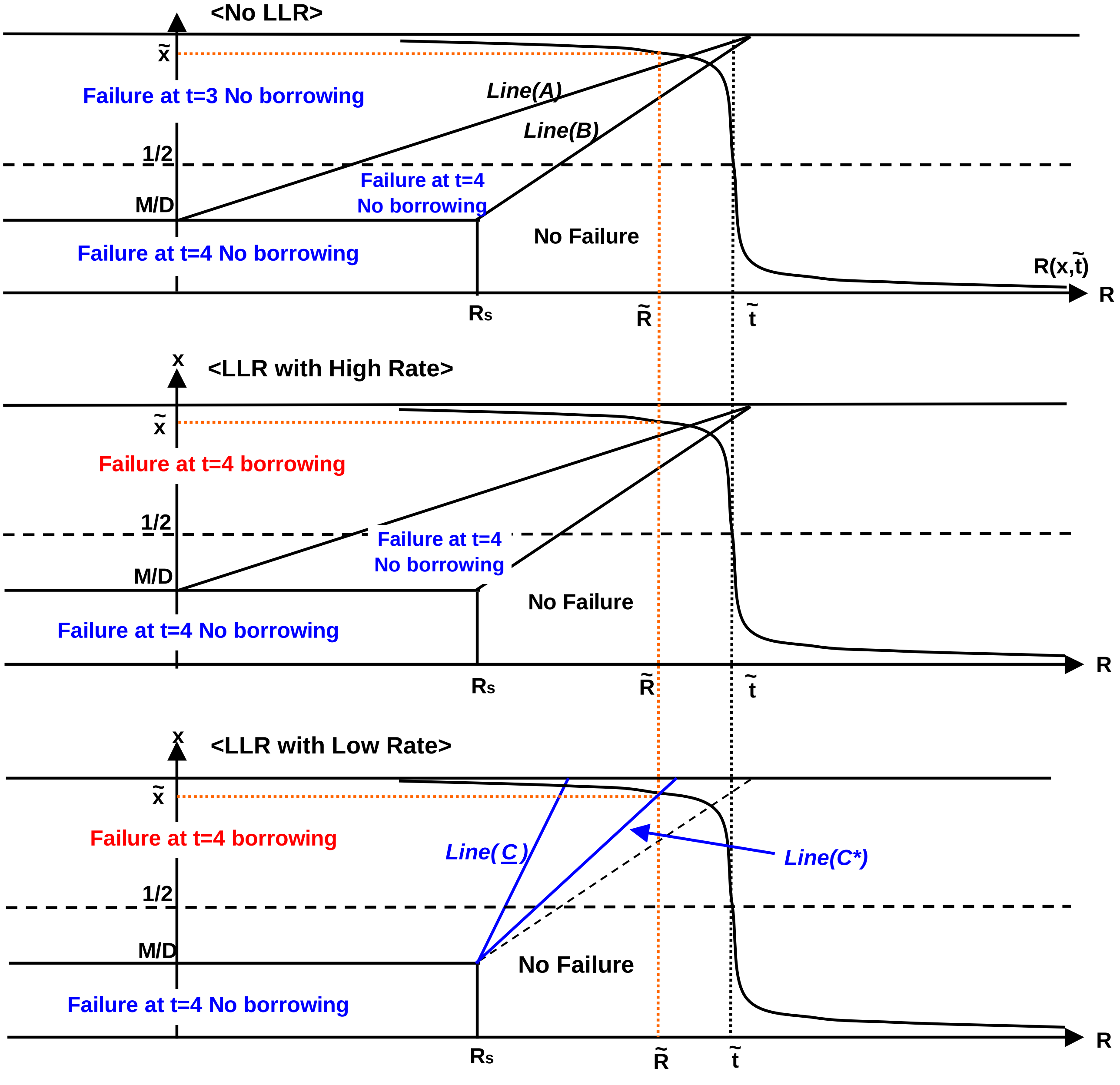

Figure 11: The PM's Best Response for a Given $\tilde{t}$ and $\tilde{x}$

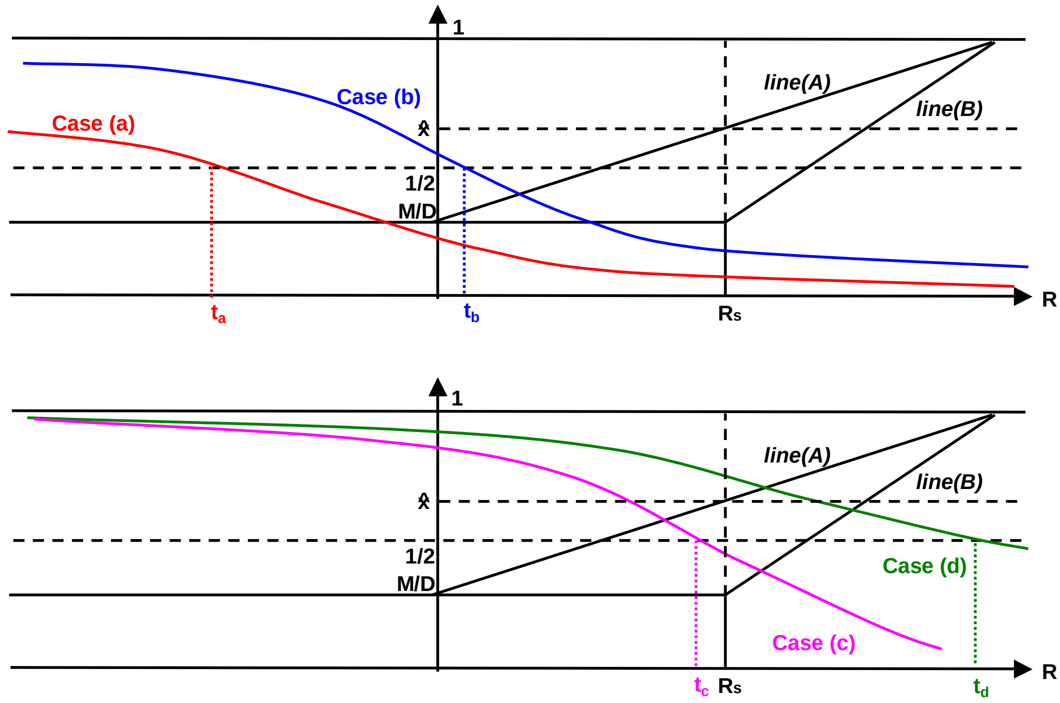

Figure 12: Four Cases of Threshold t

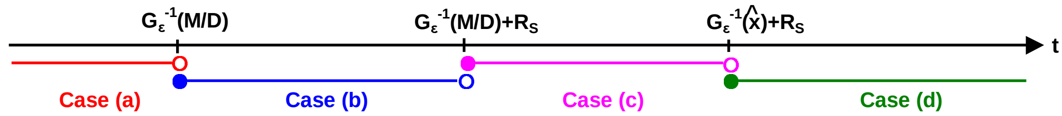

Figure 13: Four Cases of Threshold t

Case (a).(i) If $x \leq \frac{M}{D}$, then there is no difference between states for every (R, x), regardless of whether the LLR is provided at any rates or not provided at all, because the PB has enough cash for early withdrawal. Therefore, any actions of the PM are best responses. (ii) Suppose that $x > \frac{M}{D}$ (i.e., the liquidity shortage case). As $R\left(\frac{M}{D}, t\right) < 0$ and $R(x, t)$ is strictly decreasing in x, $R(x, t) < 0$. Therefore, the PM receives a payoff of zero if it provides no LLR support, whereas if it provides LLR support with any rates, its payoff is strictly negative.

From the above derivation, we establish the following remark.

Remark 3.1

Suppose that $t < G_\epsilon^{-1}\left(\frac{M}{D}\right)$. Then, the PM's best response function is:

$$A_{PM}(x) = \begin{cases} any & if \quad x \leq \frac{M}{D} \\ NL & if \quad x > \frac{M}{D}. \end{cases} \tag{9}$$

Fig.14 also illustrates the PM's best response function in (9).

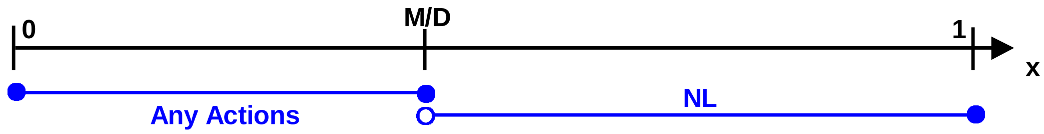

Figure 14: The PM's Best Response Function for **Case (a)**

Recall that if $x \leq \frac{M}{D}$, the PB never uses the LLR facility. Combining this observation with the second condition in (9), we conclude that in any equilibrium in which the agents' threshold strategy is of type (a), the PB never uses the LLR facility, no matter what the depositors' aggregate behavior is.

Case (b). (i) If $x \leq \frac{M}{D}$, any actions of the PM are best responses.

(ii) Suppose that $x > \frac{M}{D}$ and $(1+\lambda)\left(\frac{xD-M}{I}\right) \leq R(x,t)$. The latter condition is that the $R(x,t)$ line is to the right of Line (A) in Fig.12. Furthermore, as $R\left(\frac{M}{D},t\right) < R_s$ and $R(x,t)$ is strictly decreasing, $R(x,t) < R_s$. This means that no matter how low a lending rate r is, the PB always fails at $\tau = 4$. Therefore, while choosing NL or r with $r \geq \lambda$ gives the PM a payoff of 0, choosing r with $r < \lambda$ entails an additional loss of $-b(x) = -(xD - M) < 0$. Thus, NL and $r \geq \lambda$ are best responses.

(iii) Suppose that $x > \frac{M}{D}$ and $(1+\lambda)\left(\frac{xD-M}{I}\right) > R(x,t)$. This is the case in which the PB always fails (at $\tau = 3$ with no LLR and at $\tau = 4$ with LLR at any rates). Thus the PM's unique best response in this case is NL.

Now, we rearrange the conditions of (i)~(iii) in terms of x. Note that $(1+\lambda)\left(\frac{xD-M}{I}\right) \leq R(x,t) \Leftrightarrow (1+\lambda)Dx + IG_\epsilon^{-1}(x) \leq It + (1+\lambda)M$ and we provide the following definition:

$$\psi(x) = (1+\lambda)Dx + IG_\epsilon^{-1}(x). \tag{10}$$

Noting that $\lim_{x\to 0}\psi(x) = -\infty$, $\lim_{x\to 1}\psi(x) = \infty$, $\frac{d\psi}{dx} > 0$ and $\psi\left(\frac{M}{D}\right) = (1+\lambda)M + IG_\epsilon^{-1}\left(\frac{M}{D}\right) \leq (1+\lambda)M + It$ (the last term comes from $R\left(\frac{M}{D},t\right) \geq 0$), we obtain the following remark.

Remark 3.2

Suppose that $G_\epsilon^{-1}\left(\frac{M}{D}\right) \le t < G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s$*. Then, the PM's best response function is:*

$$A_{PM}(x) = \begin{cases} \text{any} & \text{if} \quad x \le \frac{M}{D} \\ NL \text{ or } r \text{ with } r \ge \lambda & \text{if} \quad x > \frac{M}{D} \text{ and } \psi(x) \le (1+\lambda)M + It \\ NL & \text{if} \quad \psi(x) > (1+\lambda)M + It. \end{cases} \tag{11}$$

Fig.15 illustrates the PM's best response function in **Case (b)**.

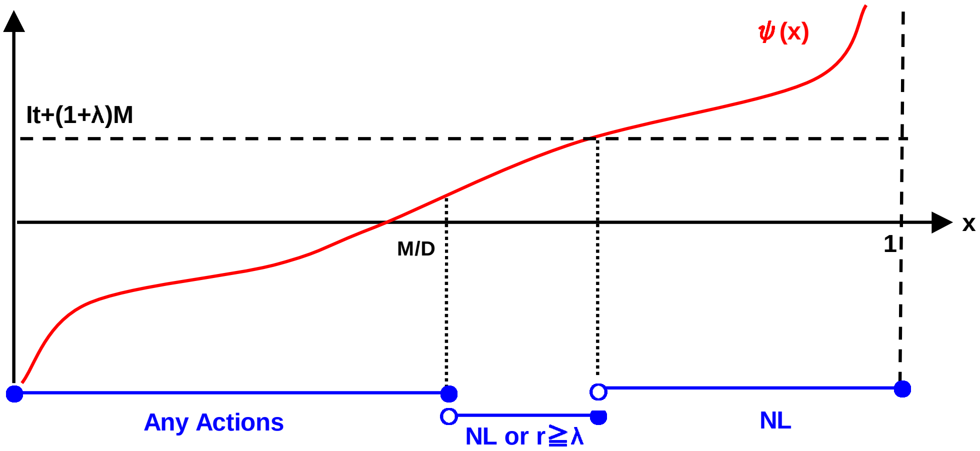

Figure 15: The PM's Best Response Function for **Case (b)**

Note that in the second condition of (11), the PB does not use the LLR facility, as also occurs in the first and the last condition. Therefore, as in the previous case, it is clear that the PB never uses the LLR in any equilibrium of type (b).

Case (c). (i) Suppose that $R_s + \lambda\left(\frac{xD-M}{I}\right) \le R(x,t)$, which implies that the $R(x,t)$ curve is to the right of Line (B) in Fig.12. In such a "solvent and liquid" case, any actions by the PM are best responses.

(ii) Suppose the following:

$$x > \frac{M}{D} \text{ and } R_s < R(x,t) < R_s + \lambda\left(\frac{xD-M}{I}\right). \tag{12}$$

This is the case where the $R(x,t)$ line lies between the vertical line of $R = R_s$ and Line (B) in Fig.12. If the PM provides no LLR or an LLR with a high rate, the PB fails at $\tau = 4$. On the other hand, if the PM provides an LLR scheme with a rate r such that:

$$R_s + r\left(\frac{xD - M}{I}\right) \leq R(x,t), \tag{13}$$

then the PB uses the LLR facility and survives (note that the left-hand side of (13) is the expression of Line (C) in Fig.7). Note that the amount of the lending money in this case is $b(x) = xD - M > 0$ and, thus, the unique maximum value of $rb(x)$, subject to (13), is attained at $r\left(\frac{xD-M}{I}\right) = R(x,t)$. That is, the unique optimal behavior is to provide an LLR scheme with the rate of $r^*(x)$ such that:

$$r^*(x) \equiv \frac{IR(x,t) - (D - M)}{xD - M}. \tag{14}$$

Note that the lending rate $r^*(x)$ in (14) is strictly positive for every x satisfying (12) because $xD - M > 0$ and the sign of the numerator is identical to that of $R(x,t) - R_s (> 0)$.

(iii) Suppose that $x > \frac{M}{D}$ and $(1+\lambda)\left(\frac{xD-M}{I}\right) \leq R(x,t) \leq R_s$. This is the case in which R on the $R(x,t)$ line lies between Line (A) and the vertical line of $R = R_s$ (see Fig.12). Thus, the PM's best response is NL or L with a high rate.

(iv) Suppose that $x > \frac{M}{D}$ and that $R(x,t) < (1+\lambda)\left(\frac{xD-M}{I}\right)$. Similarly to (iii) in **Case (b)**, the PM's unique best response is NL.

Now, we summarize (i)~(iv) in terms of x (see the Appendix). Note that $R_s + \lambda\left(\frac{xD-M}{I}\right) \leq R(x,t) \Longleftrightarrow \lambda Dx + IG_\epsilon^{-1}(x) \leq It + (1+\lambda)M - D$ and define φ as:

$$\varphi(x) = \lambda Dx + IG_\epsilon^{-1}(x). \tag{15}$$

From the arrangement in the Appendix, we establish the following remark.

Remark 3.3

Suppose that $G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s \le t < G_\epsilon^{-1}(\hat{x}) + R_s$. Then, the PM's best response function is:

$$A_{PM}(x) = \begin{cases} \text{any} & \text{if} \quad \varphi(x) \le It + (1+\lambda)M - D \\ r^*(x)\ (>0) \text{ satisfying (14)} & \text{if} \quad \varphi(x) > It + (1+\lambda)M - D \text{ and } R(x,t) > R_s \\ NL \text{ or } r \text{ with } r \ge \lambda & \text{if} \quad R(x,t) \le R_s \text{ and } \psi(x) \le It + (1+\lambda)M \\ NL & \text{if} \quad \psi(x) > It + (1+\lambda)M. \end{cases} \tag{16}$$

Fig.16 illustrates the PM's best response function in **Case (c)**.

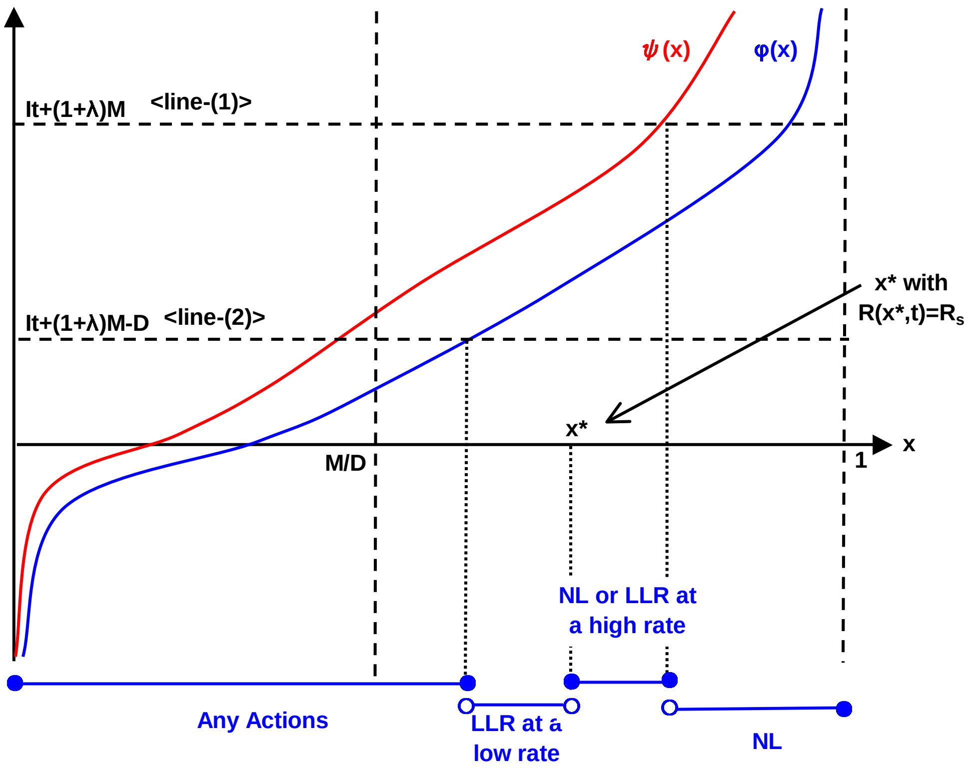

Figure 16: The PM's Best Response Function for **Case (c)**

Here, some remarks should be made. First, from (16), it is clear that the PB uses the LLR facility if and only if the second condition holds, that is, when the PB is "solvent but illiquid." Next, in this case, the lending rate is $r^*(x)$. This means that whenever the LLR facility is utilized, the optimal lending rates are strictly positive. Finally, recall that r^* is the solution to the maximization problem of $rb(x)$ subject to (13). This means that the optimal lending rate takes the highest level possible, under the restriction that the rate is low enough for solvent but illiquid banks to survive. Thus, from the PB's viewpoint, r^* can be seen as (restricted) "punitive" lending rates. It should be noted that such a punitive rate is attained via the policy maker's balance sheet soundness term $rb(x)$, whereas the restriction comes from the financial stability term. Both terms are embedded in the utility function of the policy maker. To examine the evolution of $r^*(x)$ further, we take the derivative of $r^*(x)$, as follows:

$$\begin{aligned} \frac{dr^*(x)}{dx} &= \frac{d}{dx}\left[\left(\frac{I}{xD-M}\right)\left\{t - G_\epsilon^{-1}(x) - R_s\right\}\right] && (17) \\ &= \frac{-G_\epsilon^{-1}(x)'[xD-M] - [t - G_\epsilon^{-1}(x) - R_s]D}{\{xD-M\}^2} && (18) \\ &< 0, && (19) \end{aligned}$$

because $G_\epsilon^{-1}(\cdot) > 0$, $x > \frac{M}{D}$ and $t - G_\epsilon^{-1}(x) - R_s > t - G_\epsilon^{-1}(G_\epsilon(t - R_s)) = 0$. Therefore, $r^*(x)$ is described as in Fig.17.

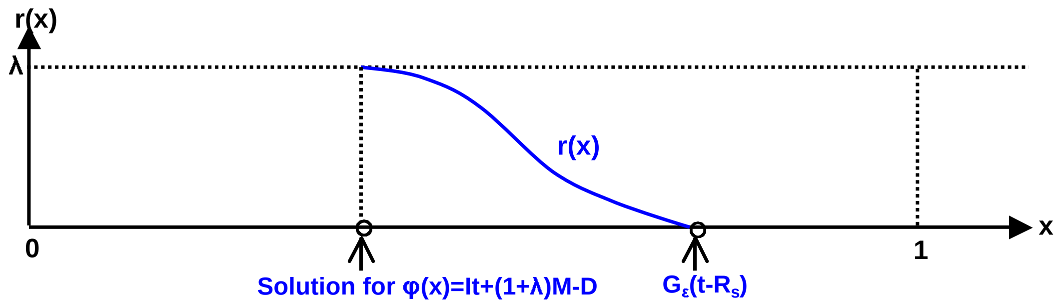

Figure 17: The Evolution of $r^*(x)$

Intuitively, as x increases, the seriousness of the "liquidity problem" faced by a solvent bank becomes greater. In such a case, the PM adjusts the punitive optimal rate so that it will not bankrupt the bank. Thus, $r(x)$ is decreasing in x.

Case (d). The only difference from **Case (c)** is that there is no x satisfying the condition of (iii). Thus, the following remark is immediately established.

Remark 3.4

Suppose that $G_{\epsilon}^{-1}(\hat{x}) + R_s \leq t$. Then, the PM's best response function is:

$$A_{PM}(x) = \begin{cases} \text{any} & \text{if} \quad \varphi(x) \leq It + (1+\lambda)M - D \\ r^*(x)\ (>0) \text{ satisfying (14)} & \text{if} \quad \varphi(x) > It + (1+\lambda)M - D \text{ and } R(x,t) > R_s \\ NL & \text{if} \quad R(x,t) \leq R_s. \end{cases} \tag{20}$$

Fig.18 illustrates the PM's best response function in **Case (d)**.

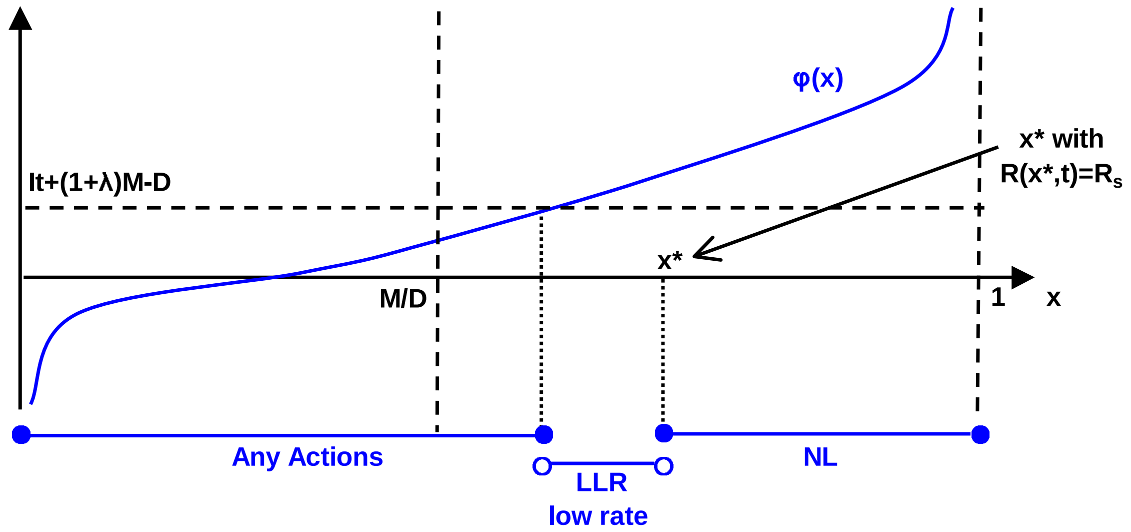

Figure 18: The PM's Best Response Function for **Case (d)**

Again, in this case, the LLR would be utilized iff the PB is solvent but illiquid, and the lending rate is strictly positive.

Now, we obtain the precise expression of the PM's best responses for all t —Remark 3.1 to Remark 3.4—and establish the following.

Theorem 3.1

In all Bayesian equilibria:

1. *whenever the PB uses the LLR scheme, the lending rate is strictly positive, and*
2. *the PB is rescued by the LLR scheme if and only if the PB is solvent but illiquid.*

Theorem 3.1 implies that *if* there exists an equilibrium categorized into **Case (c)** or **Case (d)**, then providing liquidity to solvent but illiquid banks at a positive rate is described as an equilibrium (in **Case (a)** or **Case (b)**, the LLR facility is never used).

In the following analysis, we will actually show that all equilibria are of either type (c) or (d) when β is sufficiently large. We will also examine the case where β goes to zero.

3.4 Equilibrium in the Agents' Game

Given the PM's best responses derived in the previous section, now we search for a whole equilibrium in which agents use the same threshold strategy t^*. However, before doing this, we introduce some notations and show properties that will be used in the following analysis.

First, let $R_F(t)$ be the critical R below which the PB fails, provided that all agents use the same threshold strategy t and the PM uses the best response function to such a t, derived in the previous section. This notation is borrowed from R&V and, in their model, $R_F(t)$ is increasing in t for large ts, whereas it is equal to R_s for small ts.

In our model, on the other hand, the following remark holds and, as we will see, this substantially simplifies the whole analysis.

Remark 3.5

$$R_F(t) = R_s \forall t. \tag{21}$$

In order to understand the difference, first recall that, in R&V, the shape of the (R, x) plane is always as in Fig.6. In this case, if t is large enough to satisfy the condition of Cases (c) or (d), $R_F(t)$ lies in the intersection between Line (B) and $R(x, t)$. Therefore, for such a t, $R_F(t)$ is increasing in t (see Fig.19).

On the other hand, in our model, in which the PM behaves optimally so that solvent but illiquid banks are rescued, suppose, for instance, that we have **Case (c)** (see also Fig.20). Consider a fixed $t = \tilde{t}$ that satisfies **Case (c)**. (1) Pick an R satisfying $G_\epsilon(\tilde{t} - R) < \frac{1}{\lambda D}\{IR - D + (1 + \lambda)M\}$. This means that $G_\epsilon(\tilde{t} - R)$ is below Line (B) (R_3 in Fig.20). In this case, no matter which action is taken by the PM, the PB never fails. Therefore, $R_F(t) < R$. (2) Pick an R satisfying $G_\epsilon(\tilde{t} - R) \geq \frac{1}{\lambda D}\{IR - D + (1 + \lambda)M\}$ and $R < R_s$. In this case, LLR policy is provided so that the PB survives. Therefore, $R_F(t) < R$. (3) If $R > R_s$, then the

PB always fails. Therefore, $R_F(t) > R$ and we can conclude that $R_F(\tilde{t}) = R_s$.

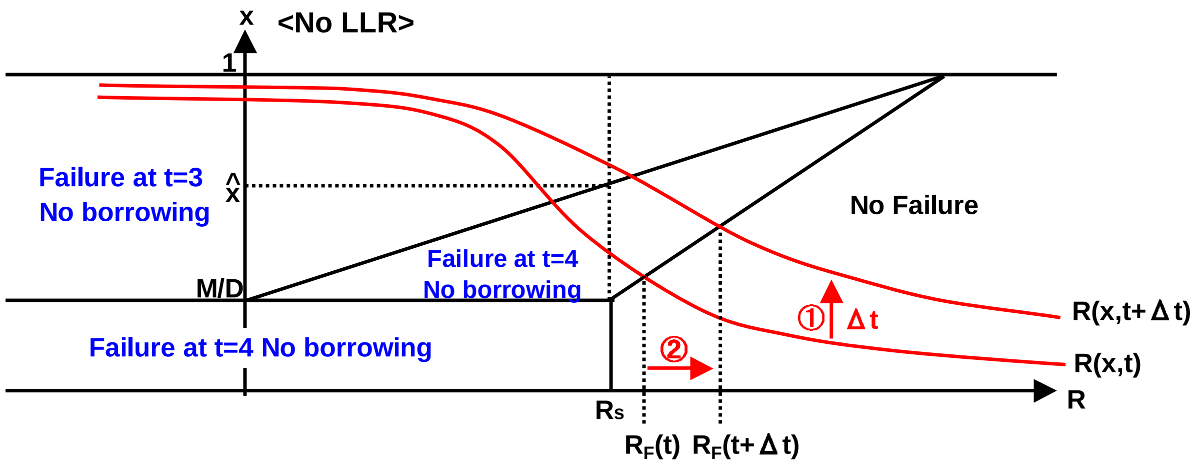

Figure 19: $R_F(t)$ in R&V: Increasing in t

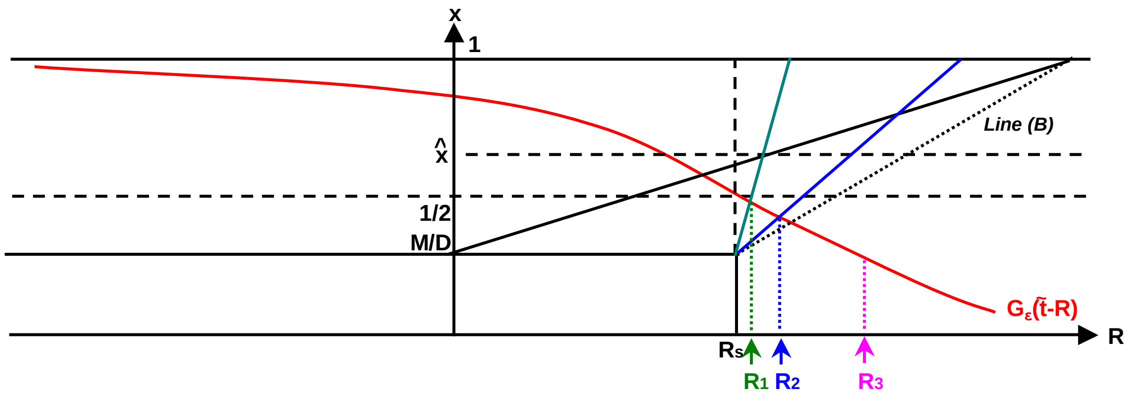

Figure 20: Derivation of $R_F(t)$ in **Case (c)**

This observation, together with the facts that: (1) the same argument holds for **Case (d)** and (2) $R_F(t) = R_s$ is obviously true for **Cases (a)** and **(b)**, shows that Remark 3.5 holds.

Second, let $P(s_i, t)$ be the (i's subjective) probability of failure, conditional on s_i and t. That is:

$$P(s_i, t) \equiv P(Failure|s_i, t) = P(R < R_F(t)|s_i) = P(R < R_s|s_i) \tag{22}$$

$$= G_{R|s_i}(R_s), \tag{23}$$

where $G_{R|s_i}(R_s)$ is *c.d.f.* of $R|s_i$. Note that:

$$R|s_i \sim N\left(\frac{\alpha\overline{R} + \beta s_i}{\alpha + \beta}, \frac{1}{\alpha + \beta}\right), \tag{24}$$

so $P(s_i, t)$ is strictly decreasing in s_i (and independent of t).

Finally, we establish the following remark.

Remark 3.6

The following holds for every t:

$$t \text{ is an equilibrium in the agents' game} \Longleftrightarrow P(t,t) = \frac{C_a}{B_a + C_a}.$$

Proof[10]

($\Longrightarrow$) Suppose that t is an equilibrium threshold. Then, for $s_i = t$ (i's observation is equal to the threshold), the i does not withdraw, so the expected payoff is zero. On the other hand, if i withdraws, its expected payoff is $P(t,t)[B_a + C_a] - C_a$. As t is an equilibrium, $P(t,t) \leq \frac{C_a}{B_a+C_a}$ must hold. If $P(t,t) < \frac{C_a}{B_a+C_a}$, then, from the continuity of $P(\cdot)$, there exists an s such that $s < t$ and $\underline{P(s,t) < \frac{C_a}{B_a+C_a}}$. For such an s, because t is assumed to be an equilibrium, W has to be a best response, which implies that $\underline{P(s,t) \geq \frac{C_a}{B_a+C_a}}$. Thus, we have a contradiction.

($\Longleftarrow$) Suppose that $P(t,t) = \frac{C_a}{B_a+C_a}$. For a given s_i and t, if i chooses W, the expected payoff is:

$$P(s_i,t)B_a - \Big(1 - P(s_i,t)\Big)C_a, \tag{25}$$

whereas i gets zero if he or she chooses NW. (1) Suppose that $s_i < t$. In this case, i uses W and its expected payoff is:

$$P(s_i,t)(B_a + C_a) - C_a > P(t,t)(B_a + C_a) - C_a = 0.$$

Thus, for $s_i < t$, W is the best response. (2) Suppose $s_i \geq t$. Then, i uses NW and, thus, its payoff is zero. On the other hand, for such an s_i, if i uses W, the expected payoff is $P(s_i,t)(B_a + C_a) - C_a \leq 0$. Thus, NW is i's best response. Therefore, if $P(t,t) = \frac{C_a}{B_a+C_a}$, the

[10] The proof is basically identical to that provided by Rochet and Vives [28].

strategy profile in which all players use the threshold strategy with t is an equilibrium in the simultaneous game among agents. ▮

We define $\eta(s) \equiv P(s, s)$. Then, from (22) and (24):

$$\eta(s) = \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta s}{\sqrt{\alpha+\beta}}\right), \tag{26}$$

where Φ is *c.d.f.* of the standard normal distribution. From Remark 3.6, t constitutes an equilibrium if and only if $\eta(t) = \frac{C_a}{B_a+C_a}$. Recall that $\eta(\cdot)$ is continuous and strictly decreasing and that $0 < \frac{C_a}{B_a+C_a} < 1$. The strictly decreasing property of η implies that if equilibria exist among the agents, then the number of the equilibria is at least one.

Now, we check the existence of an equilibrium in each case from **(a)** to **(d)**.

Case (a). Suppose that $t < G_\epsilon^{-1}(\frac{M}{D})$. Note that:

$$\lim_{s\to-\infty} \eta(s) = \lim_{k\to\infty} \Phi(k) = 1, \tag{27}$$

and

$$\lim_{s\to G_\epsilon^{-1}\left(\frac{M}{D}\right)} \eta(s) = \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}}\right). \tag{28}$$

Therefore, there exists an equilibrium in the agents' game if the following condition is satisfied (see Fig.21):

$$\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}}\right) < \frac{C_a}{B_a+C_a}. \tag{29}$$

Now, we consider the situation where $\beta \to \infty$. The limitation of the RHS of (28) is:

$$\lim_{\beta\to\infty} \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}}\right) = \lim_{\beta\to\infty} \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}} - \frac{\alpha\overline{R}}{\sqrt{\alpha+\beta}}\right) = 1. \tag{30}$$

Therefore, when $\beta \to \infty$, (29) cannot be satisfied (see also Fig.21).

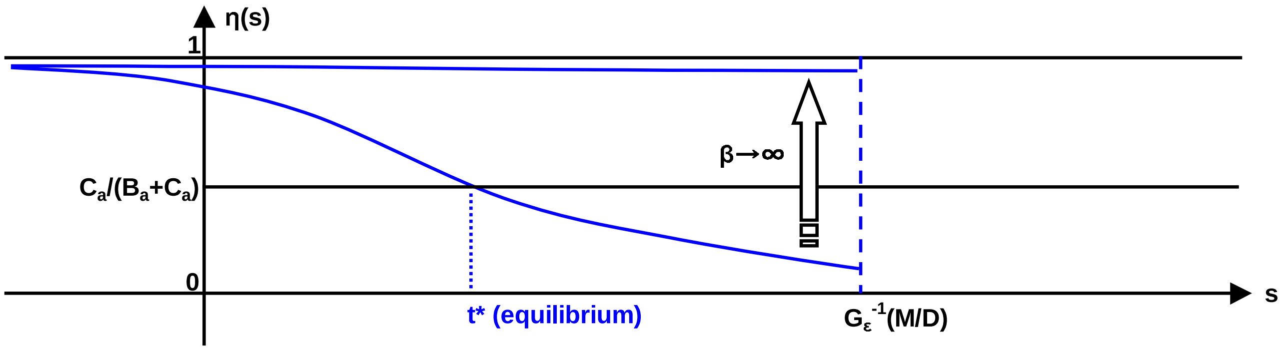

Figure 21: Existence of Equilibrium in the Agents' Game for **Case (a)**

Next, we consider the situation where $\beta \to 0$. The limitation of the RHS of (28) is:

$$\lim_{\beta \to 0} \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}}\right) = \Phi\left(\sqrt{\alpha}R_s - \frac{\alpha\overline{R}}{\sqrt{\alpha}}\right) = \Phi\left(\sqrt{\alpha}(R_s - \overline{R})\right). \tag{31}$$

Now, we establish the following theorem in **Case (a)**.

Theorem 3.2

1. *There exists $\overline{\beta}$ such that, for every β with $\beta > \overline{\beta}$, there is no Bayesian Nash equilibrium in the whole game in which $t < G_\epsilon^{-1}\left(\frac{M}{D}\right)$.*
2. *There exists $\underline{\beta}$ such that, for every β with $\beta < \underline{\beta}$, if $\Phi\left(\sqrt{\alpha}(R_s - \overline{R})\right) < \frac{C_a}{C_a+B_a}$, then there exists a Bayesian Nash equilibrium in which:*
   (a) *agents use a threshold strategy in which $t < G_\epsilon^{-1}\left(\frac{M}{D}\right)$, and*
   (b) *the PM uses a strategy satisfying (9).*

Case (b). Suppose the following:

$$G_\epsilon^{-1}\left(\frac{M}{D}\right) \le t < G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s. \tag{32}$$

Note that:

$$\eta\left(G_\epsilon^{-1}\left(\frac{M}{D}\right)\right) = \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}}\right) \tag{33}$$

$$\lim_{s\to G_\epsilon^{-1}\left(\frac{M}{D}\right)+R_s}\eta(s) = \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta\left(G_\epsilon^{-1}\left(\frac{M}{D}\right)+R_s\right)}{\sqrt{\alpha+\beta}}\right). \tag{34}$$

Thus, there exists an equilibrium in the agents' game if the following is satisfied:

$$\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta\left(G_\epsilon^{-1}\left(\frac{M}{D}\right)+R_s\right)}{\sqrt{\alpha+\beta}}\right) < \frac{C_a}{B_a+C_a} < \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}}\right). \tag{35}$$

Now, we consider the situation where $\beta\to\infty$. Similarly to the previous case:

$$\lim_{\beta\to\infty}\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}}\right) = 1. \tag{36}$$

On the other hand:

$$\lim_{\beta\to\infty}\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta\left(G_\epsilon^{-1}\left(\frac{M}{D}\right)+R_s\right)}{\sqrt{\alpha+\beta}}\right) = \lim_{\beta\to\infty}\Phi\left(\frac{\alpha}{\sqrt{\alpha+\beta}}R_s - \frac{\beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}} - \frac{\alpha\overline{R}}{\sqrt{\alpha+\beta}}\right) = 1. \tag{37}$$

Therefore, similarly to **Case (a)**, we establish the first part of Theorem 3.3 below.

Next, we consider the situation where $\beta\to 0$:

$$\lim_{\beta\to 0}\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta G_\epsilon^{-1}\left(\frac{M}{D}\right)}{\sqrt{\alpha+\beta}}\right) = \Phi\left(\sqrt{\alpha}R_s - \frac{\alpha\overline{R}}{\sqrt{\alpha}}\right) = \Phi\left(\sqrt{\alpha}(R_s-\overline{R})\right). \tag{38}$$

On the other hand:

$$\lim_{\beta\to 0}\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta\left(G_\epsilon^{-1}\left(\frac{M}{D}\right)+R_s\right)}{\sqrt{\alpha+\beta}}\right) = \Phi\left(\sqrt{\alpha}R_s - \frac{\alpha\overline{R}}{\sqrt{\alpha}}\right) = \Phi\left(\sqrt{\alpha}(R_s-\overline{R})\right). \tag{39}$$

As both limitations take the same value, the line of $\eta(s)$ never intersects with $\frac{C_a}{B_a+C_a}$ in this case.

Now, we establish the following theorem in **Case (b)**.

Theorem 3.3

1. *There exists $\overline{\beta}$ such that, for every β where $\beta > \overline{\beta}$, there is no Bayesian Nash equilibrium in the whole game in which* $G_\epsilon^{-1}\left(\frac{M}{D}\right) \leq t < G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s$.

2. *There exists $\underline{\beta}$ such that, for every β where $\beta < \underline{\beta}$, there is no Bayesian Nash equilibrium in the whole game in which* $G_\epsilon^{-1}\left(\frac{M}{D}\right) \leq t < G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s$.

Case (c). As for the third case, we have the following t:

$$G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s \leq t < G_\epsilon^{-1}(\hat{x}) + R_s \tag{40}$$

(recall that $\hat{x} = \frac{D+\lambda M}{(1+\lambda)D}$). Note that:

$$\eta\left(G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s\right) = \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left[G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s\right]}{\sqrt{\alpha+\beta}}\right) \tag{41}$$

$$\lim_{s \to G_\epsilon^{-1}(\hat{x})+R_s} \eta(s) = \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left(G_\epsilon^{-1}(\hat{x}) + R_s\right)}{\sqrt{\alpha+\beta}}\right). \tag{42}$$

Thus, there exists an equilibrium in the agents' game if the following is satisfied:

$$\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left(G_\epsilon^{-1}(\hat{x}) + R_s\right)}{\sqrt{\alpha+\beta}}\right) < \frac{C_a}{B_a + C_a} < \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left[G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s\right]}{\sqrt{\alpha+\beta}}\right). \tag{43}$$

Now, we consider the situation where β goes to ∞ and 0. First, suppose that $\beta \to \infty$. First, note that (43) is identical to the following (see the Appendix):

$$G_\epsilon^{-1}(\hat{x}) > \frac{\alpha}{\beta}(R_s - \overline{R}) - \frac{\sqrt{\alpha+\beta}}{\beta}\Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right) > G_\epsilon^{-1}\left(\frac{M}{D}\right), \tag{44}$$

and note that:

$$\lim_{\beta \to \infty}\left[\frac{\alpha}{\beta}(R_s - \overline{R}) - \frac{\sqrt{\alpha+\beta}}{\beta}\Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right)\right] = 0. \tag{45}$$

Therefore, as β goes to infinity, an equilibrium among agents exists if and only if $G_\epsilon^{-1}(\hat{x}) > 0 > G_\epsilon^{-1}\left(\frac{M}{D}\right)$. As $0 > G_\epsilon^{-1}\left(\frac{M}{D}\right)$ holds from the assumption, the first part of Theorem 3.4 below is derived.

Next, suppose that $\beta \to 0$. Note that the limitations of the left and right terms in (43) are:

$$\lim_{\beta\to 0}\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left(G_\epsilon^{-1}(\hat{x}) + R_s\right)}{\sqrt{\alpha+\beta}}\right) = \Phi\left(\sqrt{\alpha}R_s - \frac{\alpha\overline{R}}{\sqrt{\alpha}}\right) \quad (46)$$

$$\lim_{\beta\to 0}\Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left[G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s\right]}{\sqrt{\alpha+\beta}}\right) = \Phi\left(\sqrt{\alpha}R_s - \frac{\alpha\overline{R}}{\sqrt{\alpha}}\right). \quad (47)$$

In other words, both limitations take the same value. Therefore, the line of $\eta(s)$ never intersects with $\frac{C_a}{B_a+C_a}$, which implies that no equilibrium exists.

Consequently, we establish the following theorem.

Theorem 3.4

1. *There exists $\overline{\beta}$ such that for every β where $\beta > \overline{\beta}$, if $G_\epsilon^{-1}(\hat{x}) > 0$, then there exists a Bayesian Nash equilibrium in which:*
   - *(a) agents use a threshold strategy with $G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s \leq t < G_\epsilon^{-1}(\hat{x}) + R_s$, and*
   - *(b) the PM uses a strategy satisfying (16).*
2. *There exists $\underline{\beta}$ such that for every β where $\beta < \underline{\beta}$, there is no Bayesian Nash equilibrium in which $G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s \leq t < G_\epsilon^{-1}(\hat{x}) + R_s$.*

Case (d). Now, we investigate the final case. Suppose we have the following t:

$$G_\epsilon^{-1}(\hat{x}) + R_s \leq t. \quad (48)$$

Note that:

$$\eta\left(G_\epsilon^{-1}(\hat{x}) + R_s\right) = \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left[G_\epsilon^{-1}(\hat{x}) + R_s\right]}{\sqrt{\alpha+\beta}}\right) \quad (49)$$

$$\lim_{s\to\infty}\eta(s) = \lim_{k\to-\infty}\Phi(k) = 0. \quad (50)$$

Thus, there exists an equilibrium in the agents' game if the following is satisfied:

$$0 < \frac{C_a}{B_a + C_a} \le \Phi\left(\sqrt{\alpha+\beta} R_s - \frac{\alpha \overline{R} + \beta\left[G_\epsilon^{-1}(\hat{x}) + R_s\right]}{\sqrt{\alpha+\beta}} \right). \tag{51}$$

Suppose first that $\beta \to \infty$. Noting that $0 < \frac{C_a}{B_a+C_a}$ is always true from the assumption, similarly to the previous case, (51) is identical to the following:

$$G_\epsilon^{-1}(\hat{x}) \le \frac{\alpha}{\beta}(R_s - \overline{R}) - \frac{\sqrt{\alpha+\beta}}{\beta}\Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right), \tag{52}$$

and $\lim_{\beta\to\infty}\left[\frac{\alpha}{\beta}(R_s - \overline{R}) - \frac{\sqrt{\alpha+\beta}}{\beta}\Phi^{-1}\left(\frac{C_a}{B_a+C_a}\right)\right] = 0$. This establishes the first part of Theorem 3.5.

Next, suppose that $\beta \to 0$. Note that:

$$\lim_{\beta\to 0}\Phi\left(\sqrt{\alpha+\beta} R_s - \frac{\alpha \overline{R} + \beta\left(G_\epsilon^{-1}(\hat{x}) + R_s\right)}{\sqrt{\alpha+\beta}} \right) = \Phi\left(\sqrt{\alpha} R_s - \frac{\alpha\overline{R}}{\sqrt{\alpha}} \right) = \Phi\left(\sqrt{\alpha}(R_s - \overline{R}) \right). \tag{53}$$

Consequently, we establish the following theorem.

Theorem 3.5

1. *There exists $\overline{\beta}$ such that, for every β where $\beta > \overline{\beta}$, if $G_\epsilon^{-1}(\hat{x}) \le 0$, then there exists a Bayesian Nash equilibrium in which:*
   (a) *agents use a threshold strategy with $G_\epsilon^{-1}(\hat{x}) + R_s \le t$, and*
   (b) *the PM uses a strategy satisfying (20).*
2. *There exists $\underline{\beta}$ such that, for every β where $\beta < \underline{\beta}$, if $\Phi\left(\sqrt{\alpha}(R_s - \overline{R})\right) \ge \frac{C_a}{B_a+C_a}$, then there exists a Bayesian Nash equilibrium in which:*
   (a) *agents use a threshold strategy with $G_\epsilon^{-1}(\hat{x}) + R_s \le t$, and*
   (b) *the PM uses a strategy satisfying (20).*

3.5 Discussions and Implications

We can summarize all the results in the following table.

	$\beta \to \infty$	$\beta \to 0$
Case (a)	No eq.	Eq. when $\Phi\left(\sqrt{\alpha}(R_s - \overline{R})\right) < \frac{C_a}{C_a+B_a}$.
Case (b)	No eq.	No eq.
Case (c)	Eq. when $G_{\epsilon^{-1}}(\hat{x}) > 0$	No eq.
Case (d)	Eq. when $G_{\epsilon^{-1}}(\hat{x}) \leq 0$	Eq. when $\Phi\left(\sqrt{\alpha}(R_s - \overline{R})\right) \geq \frac{C_a}{C_a+B_a}$.

Consider $\beta \to \infty$ first. This is the case where the noise regarding the agents' information about a realized return is very small and every standard global game analysis takes this situation into consideration. In this case, an equilibrium of either **Case (c)** or **Case (d)** is realized, depending on the parameters of $\hat{x}$, whereas **Case (a)** and **Case (b)** never occur as an equilibrium. As we discussed, in both cases, the policy maker's optimal behavior is to help only illiquid but solvent borrowers, and the lending rates are strictly positive whenever the facility is utilized by borrowers experiencing such liquidity shortages. Furthermore, the optimal lending rates are "conditionally punitive," in the sense that they take the highest level possible under the restriction that the rate is low enough for these borrowers to survive. These results succeed in describing the LLR authority's optimal behavior as being in line with Bagehot's claims, as well as with the historical records and current operations of emergent liquidity provision. Here, it may be worth noting that, in every equilibrium, the strictly positive lending rate never exceeds the fire sale premium λ. This fact may appear inconsistent with the "penalty" rate. However, it should be recalled that λ is defined as a fire sale premium, thus λ can be interpreted as one that is realized during panics. The term "penalty" rate should be compared with rates in noncrisis periods (see Fischer [14]). Therefore, the fact that r never exceeds λ is not necessarily inconsistent with Bagehot's claim. Finally, the difference between conditions of **case (c)** and **(d)** should be mentioned. For example, $G_{\epsilon^{-1}}(\hat{x}) \leq 0$ is satisfied when M is small, which means PB is more likely to suffer from "liquidity shortage,"compared to the case of $G_{\epsilon^{-1}}(\hat{x}) > 0$. Knowing this situation, agents would choose t high enough to satisfy **(d)** in which an agent will choose to withdraw for some her signals but will choose not to withdraw for such signals under t satisfying **(c)**.

Another intriguing case is $\beta \to 0$. Such a limiting case has rarely been analyzed in the existing literature on global games but, interestingly, our model succeeds in obtaining plausible results. In this case, depending on parameters, an equilibrium of either **Case (a)** or **Case (d)** is realized. In particular, to derive certain implications for this case, it is convenient to focus on the magnitude of C_a and B_a. Consider **Case (a)** first. Note that the inequality of $\Phi\left(\sqrt{\alpha}(R_s - \overline{R})\right) < \frac{C_a}{C_a+B_a}$ is satisfied if C_a is sufficiently large compared with B_a. If an agent has no information on R and C_a is large, then it implies that the expected payoff from withdrawing is very small compared with that from not withdrawing. If so, an agent would not be likely to withdraw even if a signal is very small, which means that t is very low. Obviously, an exactly symmetric argument holds for **Case (d)**: if the benefit of withdrawing is very high and no posterior information about R is available, an agent would be likely to withdraw, which implies that the optimal t is very high. In both cases, such an agent's optimal behavior is obviously quite intuitive.

Finally, we can completely explain how the signaling role of agents' aggregate behavior works in an equilibrium and the PM infers true fundamentals. First, note that for given parameters, all of these are common knowledge among players, the PM can compute the equilibrium associated with it. Suppose, for instance, $\beta \to \infty$ and the set of parameters satisfies the condition $G_{\epsilon^{-1}}(\hat{x}) > 0$. In this case, the PM can compute the agents' optimal threshold strategy t^* as the solution of the following (see (26)):

$$\eta(s) = \Phi\left(\sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R}+\beta s}{\sqrt{\alpha+\beta}}\right) = \frac{C_a}{B_a + C_a}. \tag{54}$$

From the strictly decreasing property of $\eta(\cdot)$ and Theorem 3.4, such a t^* is uniquely determined. Given this derived t^*, the PM can compute the true fundamentals R by (8) for every observation of x.

4 Concluding Remarks

In this paper, we constructed an abstract model about emergent liquidity provision by using global game techniques in which the policy maker is an explicit player whose preference is based on the states of the borrowers as well as the soundness of its own balance sheet. Furthermore, it was assumed that the policy maker cannot distinguish solvent from insolvent borrowers ex ante and that the policy maker's decision making occurs after observing agents' aggregate behavior. With this setup, we showed that agents' aggregate behavior work as a signal to the policy maker regarding borrowers' solvency. Then, it was shown that: (1) the policy maker's optimal behavior is to help only illiquid but solvent borrowers, and (2) whenever the liquidity provision facility is utilized, optimal lending rates are strictly positive, and these rates are "conditionally punitive" in the sense that they take the highest level possible under the restriction that the rates enable solvent but illiquid borrowers to survive. Such punitive rates are attained via the policy maker's balance sheet channel embedded in its utility function.

Here it may be worth mentioning to the relationship between our results and the discussion about moral hazard problem in emergent liquidity provision contexts. The function of providing liquidity assistance has been criticized for provoking moral hazard on the borrower's side. Recalling that this problem would become severe as the borrower has an expectation that the policy maker eventually provides liquidity at a low rate or even to insolvent borrowers, our analysis suggests that the policy maker can prevent agents from having such expectation by announcing the importance of consideration for its balance sheet condition. Such announcement works as a commitment tool for the policy maker not to set the lending rate unnecessarily low because the balance sheet soundness term embedded in the policy maker's utility function motivates it to provide liquidity at "punitive" rates.

To conclude our analysis, we point out some intriguing extensions for further research.

First, we assumed that λ is an exogenous parameter. This means that λ is independent from any other variables such as R. This assumption can be justified in examining situation where R fluctuates depending on LLR user's idiosyncratic shocks which do not affect general conditions in financial markets represented by λ. On the other hand, it is also realistic to assume that λ and R are correlated. For example, a negative shock on a country's aggregated demand may trigger a small R and high premium λ simultaneously. Such generalization will be an intriguing extension of our analysis.

Second, our model assumed that the proportion of agents who withdraw their deposits is completely observable to the policy maker. However, it might also be reasonable to assume that such an observation entails some noise. This assumption could be plausible because, in some circumstances, the policy maker has to make the decision regarding whether to provide liquidity assistance within such a very short timeline that it cannot collect sufficient information about agents' exact behaviors or the borrower's daily balance sheet. For example, instead of assuming that the policy maker observes the true x, we can alternatively assume that z is an observable signal to the policy maker such that:

$$z = x + \xi_{PM}, \tag{55}$$

where ξ_{PM} is RV with, for example, $\xi_{PM} \sim N(0, 1/\gamma)$. Recall that when x entails no noise, the policy maker can predict the true realization of R. Thus, assumptions regarding the policy maker's knowledge about R do not matter. On the other hand, once we introduce noise around x, we need to impose an assumption about how precise the policy maker can directly observe the realization of R. In other words, if x does not work as a perfect signal about R to the policy maker, it would try to obtain information about R by, for example, monitoring the borrowers.

Appendix

Derivation of (16)

We summarize (i)~(iv) in terms of x. Note that $R_s + \lambda\left(\frac{xD-M}{I}\right) \leq R(x,t) \Longleftrightarrow \lambda Dx + IG_\epsilon^{-1}(x) \leq It + (1+\lambda)M - D$. We define φ as:

$$\varphi(x) = \lambda Dx + IG_\epsilon^{-1}(x). \tag{56}$$

Then $\lim_{x\to 0}\varphi(x) = -\infty$, $\lim_{x\to 1}\varphi(x) = \infty$, $\frac{d\varphi}{dx} > 0$.

Then, we show some observations. First, $\psi(x) - \varphi(x) = Dx \geq 0$ (the $\psi(x)$ curve is always above the $\varphi(x)$ curve in Fig.16). Second, $\varphi\left(\frac{M}{D}\right) = \lambda M + IG_\epsilon^{-1}\left(\frac{M}{D}\right) \leq \lambda M + I(t - R_s) = It + (1+\lambda)M - D$ (the $\varphi(x)$ curve is below line-(2) at $\frac{M}{D}$). Third, $\psi\left(\frac{M}{D}\right) = (1+\lambda)M + IG_\epsilon^{-1}\left(\frac{M}{D}\right) \leq (1+\lambda)M + I(t-R_s) = (1+\lambda)M + It - (D-M) < It + (1+\lambda)M$ and $\psi\left(\frac{M}{D}\right) - [It + (1+\lambda)M - D] = I\left(G_\epsilon^{-1}\left(\frac{M}{D}\right) - t\right) + D \leq I(-R_s) + D = M > 0$ (the $\varphi(x)$ curve is between lines-(1) and -(2) at $\frac{M}{D}$). The fifth observation is the solution for $\varphi(x) = It + (1+\lambda)M - D$, which is less than the solution for $\psi(x) = It + (1+\lambda)M$. Let the former solution be x^*. As $\psi(x)$ is strictly increasing, it suffices to show that $\psi(x^*) < It + (1+\lambda)M$. This is true because $\psi(x^*) = (1+\lambda)Dx^* + IG_\epsilon^{-1}(x^*) = Dx^* + [It + (1+\lambda)M - D] = It + (1+\lambda)M - (1-x^*)D < It + (1+\lambda)M$.

The final observation is that the solution for $R_s = t - G_\epsilon^{-1}(x)$ (equivalently, $R(x,t) = R_s$) is: **(a)** greater than the solution for $\varphi(x) = It + (1+\lambda)M - D$, and **(b)** smaller than the solution for $\psi(x) = It + (1+\lambda)M$. Let x^* be $R(x^*, t) = R_s$. From the assumption of **Case (c)**, $\frac{M}{D} < x^* < \hat{x}$ (see also Fig.12) [11]. Similarly to the above argument, it suffices to show that $\varphi(x^*) > It + (1+\lambda)M - D$

[11] From the assumption of **Case (b)**, $G_\epsilon^{-1}\left(\frac{M}{D}\right) \leq t - R_s$. On the other hand, $R(x^*, t) = R_s$ implies $G_\epsilon^{-1}(x^*) = t - R_s$. Thus, $G_\epsilon^{-1}\left(\frac{M}{D}\right) \leq G_\epsilon^{-1}(x^*)$ and because $G_\epsilon^{-1}(\cdot)$ is strictly increasing, $\frac{M}{D} < x^*$. Similarly, $G_\epsilon^{-1}(x^*) < G_\epsilon^{-1}(\hat{x})$ implies that $x^* < \hat{x}$.

for **(a)** and $\psi(x^*) < It + (1+\lambda)M$ for **(b)**. Both are true because:

$$
\begin{aligned}
\varphi(x^*) &= \lambda D x^* + I G_\epsilon^{-1}(x^*) = \lambda D x^* + I(t - R_s) = It + \lambda D x^* - D + M \\
&> It + \lambda D\left(\frac{M}{D}\right) - D + M = It + (1+\lambda)M - D \\
\varphi(x^*) &= (1+\lambda)Dx^* + IG_\epsilon^{-1}(x^*) = (1+\lambda)Dx^* + I(t-R_s) = It + (1+\lambda)Dx^* - D + M \\
&< It + (1+\lambda)D\hat{x} - D + M = It + (1+\lambda)D\left(\frac{D+\lambda M}{(1+\lambda)D}\right) - D + M = It + (1+\lambda)M.
\end{aligned}
$$

This implies (16).

Derivation of (44)

(43) is identical to:

$$
\begin{aligned}
& \sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left(G_\epsilon^{-1}(\hat{x}) + R_s\right)}{\sqrt{\alpha+\beta}} < \Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right) < \sqrt{\alpha+\beta}R_s - \frac{\alpha\overline{R} + \beta\left[G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s\right]}{\sqrt{\alpha+\beta}} \\
\iff\quad & \frac{\alpha\overline{R} + \beta\left(G_\epsilon^{-1}(\hat{x}) + R_s\right)}{\sqrt{\alpha+\beta}} > \sqrt{\alpha+\beta}R_s - \Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right) > \frac{\alpha\overline{R} + \beta\left[G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s\right]}{\sqrt{\alpha+\beta}} \\
\iff\quad & \alpha\overline{R} + \beta\left(G_\epsilon^{-1}(\hat{x}) + R_s\right) > (\alpha+\beta)R_s - \sqrt{\alpha+\beta}\Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right) > \alpha\overline{R} + \beta\left[G_\epsilon^{-1}\left(\frac{M}{D}\right) + R_s\right] \\
\iff\quad & \beta G_\epsilon^{-1}(\hat{x}) > (\alpha+\beta)R_s - \sqrt{\alpha+\beta}\Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right) - \alpha\overline{R} - \beta R_s > \beta G_\epsilon^{-1}\left(\frac{M}{D}\right) \\
\iff\quad & G_\epsilon^{-1}(\hat{x}) > \frac{\alpha+\beta}{\beta}R_s - \frac{\sqrt{\alpha+\beta}}{\beta}\Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right) - \frac{\alpha}{\beta}\overline{R} - R_s > G_\epsilon^{-1}\left(\frac{M}{D}\right) \\
\iff\quad & G_\epsilon^{-1}(\hat{x}) > \frac{\alpha}{\beta}(R_s - \overline{R}) - \frac{\sqrt{\alpha+\beta}}{\beta}\Phi^{-1}\left(\frac{C_a}{B_a + C_a}\right) > G_\epsilon^{-1}\left(\frac{M}{D}\right).
\end{aligned}
$$

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