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Application of a Search Model to Appropriate Designing of Reference Rates: Actual Transactions and Expert Judgment

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Abstract

This study, based on a search model, attempts to draw out the implications for discussions about reference rates that originated from the recent Libor manipulation scandal, with particular focus on whether the calculation of reference rates should be based solely on actual transaction data and whether the use of expert judgment should be allowed. Generally speaking, yields on financial instruments can be decomposed into elements such as risk-free rate, (credit and/or market) risk premium, and liquidity premium. The reference rate is not exceptional. In developing a model, given that in times of crisis, liquidity dried up in interbank markets where reference rates are calculated, we use a search-based asset pricing model by Duffie, Gârleanu, Pedersen (2005, 2007) to consider a situation in which market transactions are sporadic. In evaluating asset prices, we also combine this model with the robust control method, a technique for incorporating model uncertainty (e.g., a situation in which market participants lose confidence in their own pricing models and market prices during crises). The results suggest a jump in the liquidity premium to the level exceeding the risk premium based on the fundamentals, while being amplified by uncertainty. Given that reference rates are broadly used in deciding lending rates for a number of financial contracts and the prices of derivatives, it may be economically inefficient to use interest rates that include premiums which have risen due to a temporary surge in uncertainty. Thus, an expert judgment could be allowed to some extent in order to remove these premiums.

JEL Classification: G11, G12

Keywords: Reference Rate; Libor; Market Liquidity: Search Model; Robust Control.

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1. Introduction

Since the Libor (London interbank offered rate) manipulation scandal came to light, the role and governance of the reference rates in interbank markets have drawn increasing attention. In fact, active discussions have taken place among market participants, such as the British Bankers' Association (BBA), which compiles Libor, as well as national regulatory authorities and international institutions such as the International Organization of Securities Commissions (IOSCO), the Bank for International Settlements (BIS), and the Financial Stability Board (FSB), to enhance the reliability and robustness of the reference rates.

Among several topics concerning reference rates,[1] the recommendations in the Wheatley Review released by HM Treasury of the United Kingdom in September 2012 are noteworthy. The Wheatley Review reached three fundamental conclusions. First, there is a clear case in favor of comprehensively reforming Libor, rather than replacing the benchmark. Second, transaction data should be explicitly used to support Libor submissions by reporting banks. And third, market participants should continue to play a significant role in the production and oversight of Libor.

Among these three conclusions, this paper focuses mainly on the second. The process of determining the reference rates in interbank markets is known as a "fixing" arrangement.[2] We cannot deny that the problem lies in the fact that there is room for manipulation of this fixing process among banks. Therefore, as stated in the Wheatley Review, the use of transaction data must be convincing so as to leave no room for manipulation. However, the issue is not so simple. During the recent global financial crisis, not only in securitization markets with low market liquidity but also in several sovereign debt markets with ample liquidity, liquidity rapidly dried up, and transaction prices significantly deviated from the fair value measured by a pricing model. Moreover, market participants were unable to refer to either transaction prices or quoted prices. Many of them have expressed the view that growing uncertainty about a pricing

1 For information on the relationship between reference rates and financial stability or on the transmission mechanism of monetary policy, see Muto (2012), Sudo (2012), and Kawata *et al.* (2012). For a literature survey on the practical use of reference rates, see Gyntelberg and Wooldridge (2008), and Kuo, Skeie, and Vickery (2012).

2 Typically the market rate is estimated through a "fixing" arrangement, wherein an average rate is calculated from quotes contributed by a panel of banks. One of the best-known fixing arrangements is that for Libor. Compiled by the BBA, Libor refers to the interest rate at which banks in London offer to lend funds to each other just prior to 11:00 local time. The BBA collects quotes from a panel of banks. Quotes are ranked in order, the top and bottom quartiles are disregarded, and the middle two quartiles are averaged to compute Libor.

model and the value of reference assets exacerbated the liquidity problem. Interbank markets are not excepted from such a market illiquidity problem. For this reason, we can easily imagine that calculating reference rates based solely on actual transaction data may cause a problem, especially during periods of market stress.

This study develops a model to draw out the implications for the above-mentioned problem. Specifically, the model describes liquidity risk based on the search model. Moreover, combining this model with the robust control method enables us to explain the situation in which the valuation is lowered to the extent of investors' aversion to uncertainties surrounding their pricing model.

The search model attempts to explain liquidity risk with "search friction." The subject of the search is sellers' or buyers' orders that have surfaced (or their potential orders), that is, "liquidity." One major example is a model developed by Duffie, Gârleanu, and Pedersen (2005, 2007, henceforth denoted "DGP"). Under the framework of the search model, DGP (2005, 2007) explains the situation where "agents cannot find a counterparty to trade when they want to sell financial assets" with the normal asset pricing model by loosening the premise that "agents can sell financial assets whenever they want." In the recent literature, Afonso and Lagos (2012) also applied a search model to investigate the trade dynamics in the federal funds market.

The robust control method is a technique developed by L. P. Hansen and T. Sargent to concretely express the so-called Knightian uncertainty by setting a max-min problem with restrictions in the form of entropy. This method is suitable to explain the situation in which agents cannot choose one probability distribution of future uncertainty due to uncertainty over model building and parameter estimation: for example, a situation in which market uncertainty rises dramatically in times of financial crises.

Based on an intuitive conceptual model, we can decompose the reference rate into elements such as risk-free rate, (credit and/or market) risk premium, and liquidity premium. Our theoretical model indicates that during periods of market illiquidity (specifically, when market transactions are sporadic), the premiums rise while amplifying a jump in rates and a heightening of fundamental risks. Regarding the issue of whether reference rates should be based on the transaction or expert judgment,[3] given the fact that actual premiums observed in the market include various

3 Expert judgment is used in various ways in the calculation of reference rates. For example, longer-term interest rates with low transaction volume even in normal times can be obtained by interpolation using expert judgment. In times of market illiquidity when premiums rise while amplifying a jump in reference rates and deviating from fundamentals, expert judgment can be used to appropriately reflect the fundamental risks and remove these premiums. This paper deals with the second example.

miscellaneous elements (sometimes to the level exceeding the actual premiums observed in the market, as in the peripheral European sovereign market currently), an expert judgment could be allowed to the limited extent necessary to remove the liquidity premium. Given that reference rates are broadly used in deciding lending rates for a number of financial contracts and the prices of derivatives, it may not be efficient to stick rigidly with actual transaction rates, especially during periods of market illiquidity.

The remainder of this paper is organized as follows. In Section 2, we briefly sketch the basic search model based on DGP (2005, 2007). Section 3 incorporates the robust control method into the search-based asset pricing model. Section 4 provides several numerical examples to analyze the components of reference rates such as risk-free rate, fundamental risk premium, and liquidity premium. In Section 5, we conclude.

2. Basic Search Model

As for search-based asset pricing model, we basically follow the model proposed by DGP (2005, 2007). In this section, we briefly sketch the model.

We fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a filtration $\{\mathcal{F}: t \geq 0\}$ of sub-σ-algebras satisfying the usual conditions, as defined, for example, by Protter (2010). The filtration represents the flow of information commonly available to agents over time.

Agents are risk averse and infinitely lived, with a constant time-preference rate $\delta > 0$ for consumption of a single non-storable numeraire good. In our context, concrete images of agents are banks and money market funds that provide funds to banks in the interbank and money markets.

An agent can invest in a liquid risk-free security with an interest rate of r. As a form of credit constraint that rules out Ponzi schemes, the agent must enforce some lower bound on the liquid wealth process W.

Agents may trade a "risky" asset in an over-the-counter market, which means that the risky asset can be traded only bilaterally, when in contact with a counterparty. In our context, we consider a risky asset to be representative of risks in the banking sector (a typical example is Libor, and we assume assets that pay interest based on Libor).[4]

An agent is characterized by an intrinsic preference for asset ownership that is "high" or "low." A low-type agent, when owning the asset, has a holding cost (discussed later in Section 3 in a more concrete way). The agent's intrinsic type is a Markov chain,

4 As introduced in Section 1, reference rates are fixed, in practice, by using a trimmed arithmetic mean of quotes collected from banks. It is difficult to explicitly incorporate this point into the model, and thus in this paper we abstract the case in a way such that a risky asset of the banking sector is traded.

switching from low to high with intensity λ_u, and back with intensity λ_d. The intrinsic-type processes of any two agents are independent.

A fraction s of agents is initially endowed with one unit of the risky asset. An agent owns either θ_n or θ_o units of the asset, where $\theta_n < \theta_o$. For simplicity, no other positions are permitted, which entails a loss in generality.[5] Thus, the full set of agent types is $\mathcal{T} = \{ho, hn, lo, ln\}$, with "*h*" and "*l*" designating the agent's current intrinsic liquidity state as high or low, respectively, and with "*o*" or "*n*" indicating whether the agent currently owns the asset or not, respectively.

We suppose that there is a "continuum" (a non-atomic finite measure space) of agents, and let $\mu_\sigma(t)$ denote the fraction at time t of agents of type $\sigma \in \mathcal{T}$. Because the fractions of each type of agent add to 1 at any time t,

$$\mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) = 1. \tag{1}$$

Given a total supply Θ of shares per investor, market clearing requires that

$$(\mu_{lo} + \mu_{ho})\theta_o + (\mu_{ln} + \mu_{hn})\theta_n = \Theta, \tag{2}$$

which, using (1), implies that the fraction of asset owners is

$$\mu_{lo} + \mu_{ho} = s = \frac{\Theta - \theta_n}{\theta_o - \theta_n}.$$

An agent finds a counterparty with an intensity λ, reflecting the efficiency of the search technology. That is, at the successive event times of a Poisson process with some intensity parameter λ, an agent contacts another agent, chosen from the entire population "at random," meaning with a uniform distribution across the agent population.

In equilibrium, the rates of change of the fractions of the respective investor types are[6]

$$\begin{aligned}\dot{\mu}_{lo}(t) &= -2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_u\mu_{lo}(t) + \lambda_d\mu_{ho}(t)\\ \dot{\mu}_{ln}(t) &= 2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_u\mu_{ln}(t) + \lambda_d\mu_{hn}(t)\end{aligned} \tag{3}$$

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5 An appendix to the related paper (Kobayashi, Nakamura, and Ohashi [2008]) provides an additional discussion about holding positions without any limitation under the Walrasian setting. The paper can be obtained upon request.

6 The intuition for the first equation in (3) is as follows. Whenever an *lo* agent meets an *hn* agent, he sells his asset and is no longer an *lo* agent. This explains the first term on the right-hand side of (3). The second term is due to intrinsic-type changes in which lo agents become *ho* agents, and the last term is due to intrinsic-type changes from *ho* to *lo*. The other three equations can be similarly interpreted.

$$\dot{\mu}_{ho}(t) = 2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_d\mu_{ho}(t) + \lambda_u\mu_{lo}(t)$$

$$\dot{\mu}_{hn}(t) = -2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_d\mu_{hn}(t) + \lambda_u\mu_{ln}(t).$$

DGP (2005) shows that there is a unique stable steady-state solution for $\{\mu(t) : t \geq 0\}$, that is, a constant solution defined by $\dot{\mu}(t) = 0$.

Having determined the steady-state fractions of investor types, we can compute the investors' equilibrium intensities of finding counterparties of each type and, hence, their utilities for remaining lifetime consumption, as well as the Nash bargaining price of a risky asset P.

3. Introduction of Robust Control to the Search Model

In this section, we combine a basic search model with a robust control method developed by Anderson, Hansen, and Sargent (2003). To do so, we provide stochastic processes for the dividend and endowment and then introduce robust control for their parameters.

Before discussing the details of the model, we explain here the robust control method. The original concept may date back to the insight of Frank Knight, but Gilboa and Schmeidler (1989) developed an axiomatic approach of max-min expected utility theory with multiple priors in which investors' uncertainty is represented by subjective probabilities. Agents who do not know the true probability are supposed to have a set of subjective probability measures instead of a unique probability as usual, and they make decisions by optimizing the following max-min type objective function:

$$max_{\pi} min_{Q}\ U(\pi, Q).$$

In this formulation, robust strategy π improves the worst-case utility by choosing an optimal strategy within a set of admissible trading strategies. Multiplicity of prior probabilities $\{Q\}$ models the ambiguity of the likelihood of events or the model uncertainty of the pricing model. Intuitively, the robust control method is a methodology in which the best price based on the optimal strategy under the "worst-case" scenario is calculated as a conservatively estimated value, which can differ from a price based on an original reference model. More recently, Chen and Epstein (2002) provided a continuous-time intertemporal extension of the multiple-prior framework of Gilboa and Schmeidler (1989). Furthermore, Anderson, Hansen, and Sargent (2003) developed a continuous-time dynamic formulation that explains the quantitative behavior of a robust solution via a formal semi-group analysis of the corresponding Hamilton-Jacobi-Bellman (HJB) equation. Their framework fits well with portfolio choice analysis involving uncertainty about the asset and liability return process. Subsequently, many papers have been developed in this vein (e.g., Maenhout [2006]).

We assume that the wealth, dividend, and endowment processes evolve according to

D: the cumulative dividend process

$$dD_t = m_D dt + \sigma_D dB_{1,t}, \tag{4}$$

e: the cumulative endowment process

$$de_t = m_e dt + \sigma_e \left(\rho_i dB_{1,t} + \sqrt{1-\rho_i{}^2} dB_{2,t}\right)$$
$$\rho_i = \begin{cases} \rho_h & if \;\; high\; type \\ \rho_l & if \;\; low\; type \end{cases} \tag{5}$$
$$\rho_h < \rho_l, \text{ and}$$

W: the wealth process

$$dW_t = (rW_t - c_t)dt + \theta_t dD_t + de_t - Pd\theta_t$$
$$\theta_t = \begin{cases} \theta_o & if \;\; lo/ho\; type \\ \theta_n & if \;\; ln/hn\; type \end{cases}, \tag{6}$$

where coefficients are all constants, and $B_{1,t}$ *and* $B_{2,t}$ are a standard Brownian motion with respect to the given probability space and filtration ($\mathcal{F}_t$) and are independent from each other. *c* is consumption and *P* is the price of a risky asset (which is a constant in equilibrium that we examine in this section). We suppose that the cumulative dividend process of the asset represents the risk of the banking sector, and the purchase of this asset by an agent means the provision of funds to the banking sector. In our context, risk can be interpreted both as market and credit risk in the banking sector, both of which are related to fundamentals.

An agent *i* whose intrinsic type is currently high (that is, with $\rho_i(t) = \rho_h$) values the asset more highly than does a low-intrinsic-type agent, because the increments of the high-type endowment have a lower conditional correlation with the asset's dividends. In short, we assume that an agent takes risks in the banking sector by taking account of the correlation with the endowment received separately by itself.

We assume that there exists model uncertainty with regard to the dividend process and search intensity λ.[7] To explicitly represent model uncertainty, now let us introduce the set of prior probability measures Q^{φ} on $(\Omega, \mathcal{F})$. We assume that all measures in Q^{φ} are equivalent to $\mathcal{P}$, which is defined in the usual way, that is, via the density generators by Girsanov's theorem. As our model includes both the diffusion process and the Poisson process, we need the Doléans-Dade stochastic exponentials to relate a prior measure to a reference measure.

Prior probability measures Q^{φ} on $(\Omega, \mathcal{F})$ are defined by

$$\left.\frac{dQ^{\varphi}}{dp}\right|_{F_t} = Z_t^{Q^{\varphi}} = \mathcal{E}(\phi, B_1)_t \mathcal{E}(\eta, \widetilde{N})_t, \tag{7}$$

where a density generator $\varphi = (\phi, \eta)$ belongs to the set of all progressively measurable processes. The Doléans-Dade stochastic exponentials, $\mathcal{E}(\phi, B_1)_t$ and $\mathcal{E}(\eta, \widetilde{N})_t$, are given by

$$\begin{aligned} \mathcal{E}(\phi, B_1)_t &= exp\left(\int_0^t \phi_s dB_{1,s} - \frac{1}{2}\int_0^t \phi_s^2 ds\right) \\ \mathcal{E}(\eta, \widetilde{N})_t &= exp\left(\int_0^t \int_R ln\big(1+\eta(z)\big) d\widetilde{N}(ds, dz) \right. \\ &\qquad \left. - \int_0^t \int_R \Big(\eta(z) - ln\big(1+\eta(z)\big)\Big) \nu_E(dz)ds\right) \end{aligned} \tag{8}$$

where $\widetilde{N}$, and ν_E are the compensated Poisson process and its jump measure, respectively.[8] See, for example, Cont and Tankov (2004) regarding the change of measure and entropy representation for the Levy processes, which include a Poisson process.

We assume that agents have constant-absolute-risk-averse (CARA) additive utility U, with a coefficient of absolute risk aversion γ. Let δ be a discount rate and discounted relative entropy be $\mathcal{R}_t(p\|Q) = E^p\left[e^{-\delta(T-t)}\frac{dQ}{dp} ln\frac{dQ}{dp}\right] = E^Q\left[e^{-\delta(T-t)} ln\frac{dQ}{dp}\right]$. Let J be the value function, and our problem to solve is a robust utility optimization problem with a relative entropy constraint, which is given by

$$J(w, \sigma) = sup_c inf_Q E_t^{Q^{\varphi}}\left[\int_t^{\infty} e^{-\delta(s-t)} U(c_s) ds + \omega \int_t^{\infty} e^{-\delta(s-t)} lnZ_{t,s} ds\right]. \tag{9}$$

The above sup-inf representation is the core of robust control problem formulation,

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7 Of course, the introduction of model uncertainty with regard to the endowment process and intrinsic-type-change intensities λ_u, λ_d is possible. However, the most fundamental parameters for investors in our model are the dividend process and search intensity, which we highlight in this paper.

8 Under Q, $B_{1,t}^Q = B_{1,t} - \int_0^t \phi_s ds$ is a Brownian motion, and $(1+\eta)\lambda$ is new intensity for jumps.

and ω (> 0) is a control parameter of the agent's uncertainty aversion[9]. Note that the penalty parameter ω of (9) plays a role in measuring the strength of the investor's preference for robustness. An agent with lower ω is more uncertainty averse. On the contrary, $\omega \to \infty$ corresponds to the normal utility maximization problem.

For each investor type σ ($\in \{lo, ln, ho, hn\}$), the HJB equation is explicitly written as

$$0 = sup_c inf_Q \left[\partial_t J(w,\sigma) + U(c_t) + \mathcal{A}^w J(w,\sigma) + \mathcal{B}^w J(w,\sigma) - \delta J(w,\sigma) + \frac{\omega}{2}\phi^2 - \omega\{\eta - (1+\eta)ln(1+\eta)\}\lambda \cdot 1_{\{\sigma=lo,hn\}}\right],$$

where

$$\begin{aligned}
\mathcal{A}^w J(w,lo) &= (rw - c + \theta_o m_D + \theta_o \sigma_D \phi + m_e)\partial_w J \\
&\quad + \frac{1}{2}(\theta_o^2\sigma_D^2 + \sigma_e^2 + 2\rho_l \theta_o \sigma_D \sigma_e)\partial_{ww} J \\
\mathcal{A}^w J(w,hn) &= (rw - c + \theta_n m_D + \theta_n \sigma_D \phi + m_e)\partial_w J \\
&\quad + \frac{1}{2}(\theta_n^2\sigma_D^2 + \sigma_e^2 + 2\rho_h \theta_n \sigma_D \sigma_e)\partial_{ww} \\
\mathcal{A}^w J(w,ln) &= (rw - c + \theta_n m_D + \theta_n \sigma_D \phi + m_e)\partial_w J \\
&\quad + \frac{1}{2}(\theta_n^2\sigma_D^2 + \sigma_e^2 + 2\rho_l \theta_n \sigma_D \sigma_e)\partial_{ww} J \\
\mathcal{A}^w J(w,ho) &= (rw - c + \theta_o m_D + \theta_o \sigma_D \phi + m_e)\partial_w J \\
&\quad + \frac{1}{2}(\theta_o^2\sigma_D^2 + \sigma_e^2 + 2\rho_h \theta_o \sigma_D \sigma_e)\partial_{ww} J. \\
\mathcal{B}^w J(w,lo) &= \lambda_u[J(w,ho) - J(w,lo)] \\
&\quad + 2(1+\eta)\lambda\mu_{hn}[J(w + P(\theta_o - \theta_n), ln) - J(w,lo)] \\
\mathcal{B}^w J(w,hn) &= \lambda_d[J(w,ln) - J(w,hn)] \\
&\quad + 2(1+\eta)\lambda\mu_{lo}[J(w - P(\theta_o - \theta_n), ho) - J(w,hn)] \\
\mathcal{B}^w J(w,ln) &= \lambda_u[J(w,hn) - J(w,ln)] \\
\mathcal{B}^w J(w,ho) &= \lambda_d[J(w,lo) - J(w,ho)].
\end{aligned} \tag{10}$$

As for the infimum part of the HJB equation for the *lo*-type investor, the minimum

9 As for the robust control method, there are several versions. One origin from variety emerges is a way of restricting the coverage of the "worst-case" scenario, as the literally worst-case scenario can go too far (e.g., a disaster can occur), rendering decision making meaningless. In order to delete meaningless scenarios and maintain an economic intuition at the same time, Andersen, Hansen, and Sargent (2003) proposed the use of "relative entropy" (also known as "Kullback-Leibler distance") as a definition of distance from the reference model.

point is attained at

$$\phi_t^* = -\frac{\theta_o \sigma_D \partial_w J}{\omega} \tag{11}$$

$$\eta_t^* = exp\left(-\frac{2\mu_{hn}[J(w+P(\theta_o-\theta_n),ln)-J(w,lo)]}{\omega}\right) - 1 \ =:\ exp\left(-\frac{2\mu_{hn}\Lambda_{P,lo}}{\omega}\right) - 1. \tag{12}$$

Because the target function is convex with respect to both ϕ and η, each of the above solutions (ϕ_t^*, η_t^*) is the global minimum.

Similarly, for the *hn*-type investor, the minimum point is attained at

$$\phi_t^* = -\frac{\theta_n \sigma_D \partial_w J}{\omega}$$

$$\eta_t^* = exp\left(-\frac{2\mu_{lo}[J(w-P(\theta_o-\theta_n),ho)-J(w,hn)]}{\omega}\right) - 1 \ =:\ exp\left(-\frac{2\mu_{lo}\Lambda_{P,hn}}{\omega}\right) - 1,$$

for the *ln*-type investor the minimum point is attained at

$$\phi_t^* = -\frac{\theta_n \sigma_D \partial_w J}{\omega},$$

and for the *ho*-type investor the minimum point is attained at

$$\phi_t^* = -\frac{\theta_o \sigma_D \partial_w J}{\omega} \ .$$

Therefore, by imputing the above minimum point into the HJB equations, we can rewrite the HJB equations as follows.

$$\begin{aligned} 0 = sup_c[&\partial_t J(w,lo) + U(c_t) - \delta J(w,lo) \\ &+ (rw - c + \theta_o m_D + m_e)\partial_w J + \frac{1}{2}(\theta_o^2\sigma_D^2 + \sigma_e^2 + 2\rho_l\theta_o\sigma_D\sigma_e)\partial_{ww} J \\ &- \frac{\theta_o^2\sigma_D^2}{2\omega}(\partial_w J)^2 + \lambda\left[2\mu_{hn}\Lambda_{P,lo}e^{-\frac{2\mu_{hn}\Lambda_{P,lo}}{\omega}} + \omega\left\{1 - e^{-\frac{2\mu_{hn}\Lambda_{P,lo}}{\omega}}\left(1 + \frac{2\mu_{hn}\Lambda_{P,lo}}{\omega}\right)\right\}\right] \\ &+ \lambda_u\Lambda_{lo} \end{aligned}$$

for the *lo*-type investor.

For other types of investors, we can obtain similar equations.

To continue to calculate equations explicitly, we apply the CARA utility as below;

$$U(c) = -e^{-\gamma c}. \tag{13}$$

By following DGP (2007), we can guess the value function to solve the HJB equations as follows:

$$J(w,\sigma) = -e^{-r\gamma(w + a_\sigma + \bar{a})}. \tag{14}$$

The optimal consumption is determined by $\partial_c U(c) - \partial_w J = 0$, that is,

$$c^* = (\partial_c U)^{-1}(\partial_w J) = -\frac{\log r}{\gamma} + r(w + a_\sigma + \bar{a}),$$

where $\overline{a} = \frac{1}{r}\left(\frac{\log r}{\gamma} + m_e - \frac{1}{2} r\gamma\sigma_e^2 - \frac{r-\delta}{r\gamma}\right)$, and constant a_σ is calculated numerically later.

As argued in Maenhout (2004), to preserve the homotheticity (or independence of an impact on parameter from the wealth level) of the value function, the robustness parameter ω of the relative entropy term should be

$$\omega = \omega_0 J(w, \sigma) \tag{15}$$

for some constant $\omega_0 < 0$.[10]

Based on the above setting, we then try to derive HJB equations. As for the *lo*-type investor, the HJB equation can be rewritten as follows.

$$0 = -r^2\gamma a_{lo} + r\gamma\theta_o m_D - \frac{1}{2} r^2\gamma^2(\theta_o^2\sigma_D^2 + 2\rho_l\theta_o\sigma_D\sigma_e) + \frac{\theta_o^2\sigma_D^2}{2\omega_0} r^2\gamma^2$$

$$+ \lambda\left[2\mu_{hn}\left(-e^{-r\gamma(P(\theta_o-\theta_n)+a_{ln}-a_{lo})} + 1\right)e^{\frac{2\mu_{hn}\left(-e^{-r\gamma(P(\theta_o-\theta_n)+a_{ln}-a_{lo})}+1\right)}{\omega_0}}\right.$$

$$\left. -\omega_0 \cdot \left\{1 - e^{\frac{2\mu_{hn}\left(-e^{-r\gamma(P(\theta_o-\theta_n)+a_{ln}-a_{lo})}+1\right)}{\omega_0}}\left(1 - \frac{2\mu_{hn}\left(-e^{-r\gamma(P(\theta_o-\theta_n)+a_{ln}-a_{lo})}+1\right)}{\omega_0}\right)\right\}\right]$$

$$+ \lambda_u\left(-e^{-r\gamma(a_{ho}-a_{lo})} + 1\right).$$

For other types of investors, we can obtain similar equations.

As in DGP (2007), the equilibrium price P is determined using Nash bargaining with seller bargaining power q. However, we do not go far into the issue of bargaining power here, and it is given exogenously. (In reality and during times of crisis, while lenders of funds can find borrowers other than banks, borrowers find themselves at a loss in the absence of access to loan funds. Thus, bargaining power may be asymmetric, but we leave this to future work.) The bargaining price satisfies

$$a_{lo} - a_{ln} \le P(\theta_o - \theta_n) \le a_{ho} - a_{hn}.$$

10 Because we apply "negative" exponential utility, we take ω_0 as negative values to keep original parameter ω positive. A "homothetic" penalty parameter is useful due to not only analytical tractability but also economic interpretation. See Maenhout (2004) for more details.

By summing up, we obtain the following result.

In equilibrium, an agent's consumption rate is given by

$$c^* = -\frac{\log r}{\gamma} + r(w + a_\sigma + \bar{a}) \text{ , where } \bar{a} = \frac{1}{r}\left(\frac{\log r}{\gamma} + m_e - \frac{1}{2} r\gamma\sigma_e^2 - \frac{r-\delta}{r\gamma}\right),$$

the value function is given by

$$J(w,\sigma) = -e^{-r\gamma(w + a_\sigma + \bar{a})},$$

and $(a_{lo}, a_{ln}, a_{ho}, a_{hn}, P) \in \mathbb{R}^5$ solves

$$0 = -r^2\gamma a_{lo} + r\gamma\theta_o m_D - \frac{1}{2} r^2\gamma^2(\theta_o^2\sigma_D^2 + 2\rho_l\theta_o\sigma_D\sigma_e) + \frac{\theta_o^2\sigma_D^2}{2\omega_0} r^2\gamma^2$$

$$+\lambda\left[2\mu_{hn}\left(-e^{-r\gamma(P(\theta_o-\theta_n)+a_{ln}-a_{lo})}+1\right)\cdot e^{\frac{2\mu_{hn}\left(-e^{-r\gamma(P(\theta_o-\theta_n)+a_{ln}-a_{lo})}+1\right)}{\omega_0}}\right.$$

$$\left.-\omega_0\cdot\left\{1 - e^{\frac{2\mu_{hn}\left(-e^{-r\gamma(P(\theta_o-\theta_n)+a_{ln}-a_{lo})}+1\right)}{\omega_0}}\cdot\left(1 - \frac{2\mu_{hn}\left(-e^{-r\gamma(P(\theta_o-\theta_n)+a_{ln}-a_{lo})}+1\right)}{\omega_0}\right)\right\}\right]$$

$$+\lambda_u\left(-e^{-r\gamma(a_{ho}-a_{lo})}+1\right),$$

for the *lo*-type investor,

$$0 = -r^2\gamma a_{hn} + r\gamma\theta_n m_D - \frac{1}{2} r^2\gamma^2\left(\theta_n^2\sigma_D^2 + 2\rho_h\theta_n\sigma_D\sigma_e\right) + \frac{\theta_n^2\sigma_D^2}{2\omega_0} r^2\gamma^2 \quad (16)$$

$$+\lambda\left[\begin{array}{l} 2\mu_{lo}\left(-e^{-r\gamma(-P(\theta_o-\theta_n)+a_{ho}-a_{hn})}+1\right)\cdot e^{\frac{2\mu_{lo}\left(-e^{-r\gamma(-P(\theta_o-\theta_n)+a_{ho}-a_{hn})}+1\right)}{\omega_0}} \\ -\omega_0\left\{1 - e^{\frac{2\mu_{lo}\left(-e^{-r\gamma(-P(\theta_o-\theta_n)+a_{ho}-a_{hn})}+1\right)}{\omega_0}}\left(1 - \frac{2\mu_{lo}\left(-e^{-r\gamma(-P(\theta_o-\theta_n)+a_{ho}-a_{hn})}+1\right)}{\omega_0}\right)\right\}\end{array}\right]$$

$$+\lambda_d\left(-e^{-r\gamma(a_{ln}-a_{hn})}+1\right)$$

for the *hn*-type investor,

$$0 = -r^2\gamma a_{ln} + r\gamma\theta_n m_D - \frac{1}{2} r^2\gamma^2(\theta_n^2\sigma_D^2 + 2\rho_l\theta_n\sigma_D\sigma_e) + \frac{\theta_n^2\sigma_D^2}{2\omega_0} r^2\gamma^2$$

$$+\lambda_u\left(-e^{-r\gamma(a_{hn}-a_{ln})}+1\right)$$

for the *ln*-type investor, and

$$0 = -r^2\gamma a_{ho} + r\gamma\theta_o m_D - \frac{1}{2} r^2\gamma^2(\theta_o^2\sigma_D^2 + 2\rho_h\theta_o\sigma_D\sigma_e) + \frac{\theta_o^2\sigma_D^2}{2\omega_0} r^2\gamma^2$$

$$+\lambda_d\left(-e^{-r\gamma(a_{lo}-a_{ho})}+1\right)$$

for the *ho*-type investor,

as well as the Nash bargaining equation

$$q\left(-e^{r\gamma\left(P(\theta_o-\theta_n)-(a_{lo}-a_{ln})\right)}+1\right)=(1-q)\left(-e^{r\gamma\left(-P(\theta_o-\theta_n)+(a_{ho}-a_{hn})\right)}+1\right), \tag{17}$$

where $q \in (0,1)$ is the bargaining power of the seller of a risky asset.

These equations are highly nonlinear, however, it is possible to calculate them numerically. By following a procedure similar to DGP (2007), we can confirm our candidate optimal consumption and trading strategy that satisfies the transversality condition.

A natural benchmark is the limit price associated with vanishing search frictions, characterized as follows.

If the fraction of asset owner (see equation(2)) $s < \mu_{ho} + \mu_{hn}$[11], then, as $\lambda \to \infty$,

$$P \to \frac{m_D - \frac{1}{2}r\gamma\left((\theta_o+\theta_n)\sigma_D^2 + 2\rho_h\sigma_D\sigma_e\right)}{r} + \frac{\sigma_D^2\gamma(\theta_o+\theta_n)}{2\omega_0}. \tag{18}$$

For reference, the non-robust HJB equations correspond to the limit case; $\omega_0 \to -\infty$. In this case,

$$P \to \frac{m_D - \frac{1}{2}r\gamma\left((\theta_o+\theta_n)\sigma_D^2 + 2\rho_h\sigma_D\sigma_e\right)}{r}. \tag{19}$$

This is exactly the same as the result in DGP (2007). To state it differently, as long as model uncertainty exists, the limit price does not converge to the price with vanishing search frictions. If an investor is uncertainty averse, he evaluates the asset price at a level lower by $\frac{\sigma_D^2\gamma(\theta_o+\theta_n)}{2\omega_0}(<0)$ than the non-robust limit price.

4. Numerical Results

In this section, we give some numerical examples.

We select the same parameters as the DGP (2007) setting to compare both results and highlight our "uncertainty aversion" effect. Table 1 contains the exogenous search model parameters for the base-case and resulting steady-state fractions of each type. Table 2 contains additional base-case parameters with risk aversion and uncertainty aversion.

11 This condition implies $\mu_{lo} < \mu_{hn}$. As the amount of risk assets in the market is exogenously given, the market is rationed depending on the steady-state fraction of agents. If a sign of inequality is reversed, a different result is obtained by following a similar calculation here.

λ	λ_u	λ_d	s	q
625	5	0.5	0.8	0.5

μ_{ho}	μ_{hn}	μ_{lo}	μ_{ln}
0.7972	0.1118	0.0028	0.0882

Table 1: Base-case parameters for search model and steady-state masses

Note: Search intensity λ = 625 implies that an agent expects to be in contact with 2λ = 1,250 other agents each year, that is, 1,250/250 = 5 agents a day. Given the equilibrium mass of potential buyers, the average time needed to find a counterparty is $250 \times (2\mu_{hn})^{-1}$ = 1.8 days.

r, δ	γ	ρ_h	ρ_l	μ_e	σ_e	μ_D	σ_D	θ_o	θ_n	ω_0
0.05	0.01	-0.5	0.5	10000	10000	1	0.5	20000	0	-10, -50

Table 2: Additional parameters with risk aversion and uncertainty aversion

Note: The intrinsic type of an agent is identified with correlation parameter $\rho_l > \rho_h$. A high-type agent values the asset more highly than does a low-type agent, because the increments of the low-type endowment have lower conditional correlation.
An agent owns either $\theta_o > \theta_n$ units of the asset. No other positions are permitted.
Drift and volatility parameters for dividends and endowment are just examples.
ω_0 (< 0) is a penalty parameter that controls the agent's uncertainty aversion. $\omega_0 \to -\infty$ corresponds to a non-robust agent who believes his own reference model. On the contrary, as ω_0 approaches zero, an agent becomes more uncertainty averse.

Chart 1: Relationship between search intensity and transaction rate

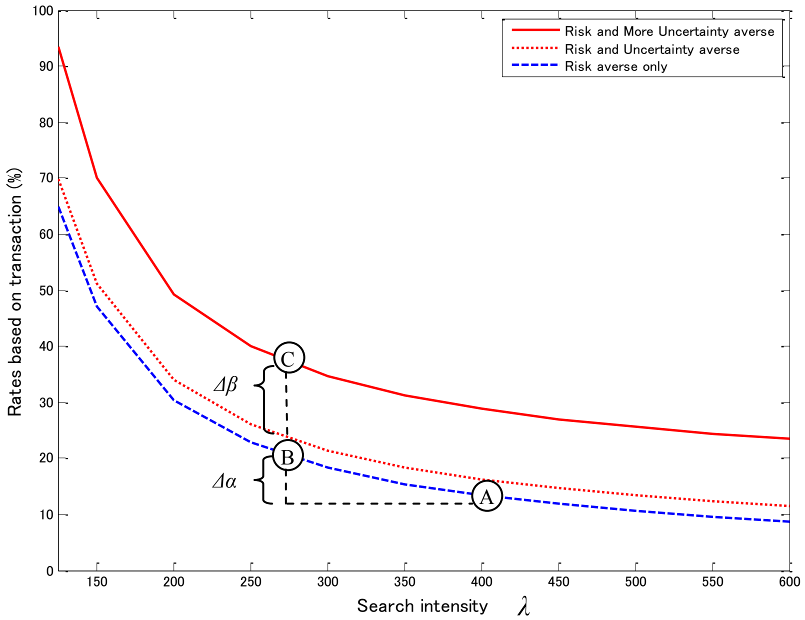
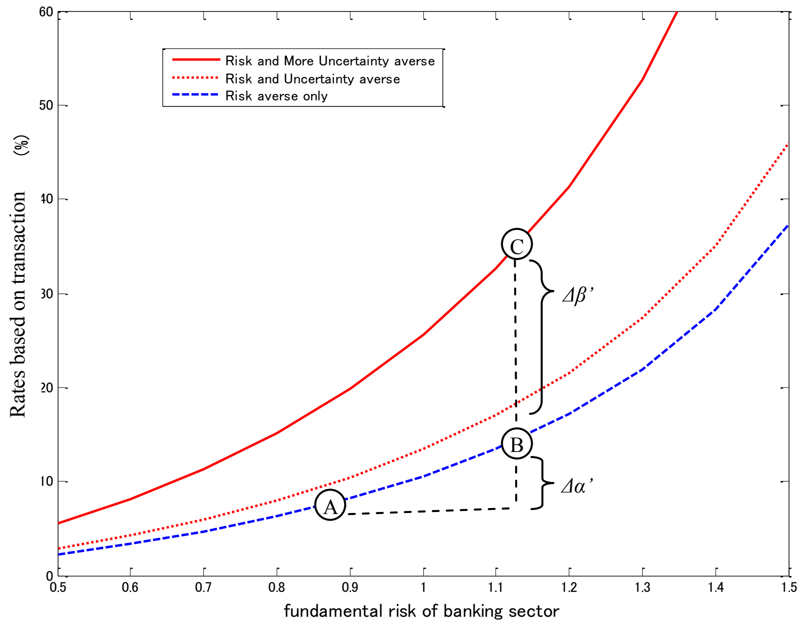


Chart 1 shows that the relationship between search parameters and reference rates. The further to the left on the horizontal axis, the more difficult it is to find counterparties to trade with. The dotted blue line assumes a risk-averse agent, as in DGP (2007). For such an agent, the harder it is to find a counterparty in the market (i.e., the lower the market liquidity), the higher the transaction rate. The dotted red line assumes a risk-averse and uncertainty-averse agent, and suggests that the heightened uncertainty has pushed up the premium (similarly, the solid red line assumes a risk-averse and "more" uncertainty-averse agent). Now let us assume that *A* represents a condition before the market becomes volatile. If the volume of interbank market transactions shrinks rapidly due to a large shock and a bank is a risk-averse agent, *A* will move, for example, to *B* reflecting the low liquidity (the transaction rate will rise by $\Delta\alpha$). If the market conditions are extremely uncertain, a bank can be not only risk averse but also uncertainty averse. In this case, *A* will move, for example, to *C* (the transaction rate will rise by $\Delta\alpha + \Delta\beta$). Although it depends on the definition of reference rates, if we define reference rates as just a tool to measure the degree of "risk-free rate + (market and/or credit) risk premium related to the banking sector" excluding the liquidity premium related to increased difficulty in finding a counterparty ($\Delta\alpha$) and premiums heightened by increased uncertainty ($\Delta\beta$), we can say the following: *C* is the actual transaction rate, but it contains a number of factors other than the fundamental risk premium; and to see the rate that only reflects the fundamental risk premium in the banking sector, an expert judgment process is necessary to exclude $\Delta\alpha + \Delta\beta$.[12]

Chart 2: Relationship between fundamental risk and transaction rate



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12 Strictly speaking, *A* already includes some liquidity premium. It can be said that, in reality, *A* moves to *B* or *C* (i.e., faces increased difficulty in finding a counterparty) against the background of the heightened fundamental risk. It is necessary to practice decomposing reference rates into their elements on a routine basis. This is important in establishing method for appropriate expert judgment.

Chart 2 shows that the relationship between reference rates and fundamental risk in the banking sector as a function of a volatility scaling of σ_D. The further to the left on the horizontal axis, the higher the fundamental risk of the banking sector as a whole. The chart suggests the usual price assessment in which high risk corresponds to high return. The dotted blue line assumes a risk-averse agent, as in DGP (2007). The dotted red line assumes a risk-averse and uncertainty-averse agent, and suggests that the heightened uncertainty has pushed up the premium (similarly, the solid red line assumes a risk-averse and "more" uncertainty-averse agent). Now let us assume that *A* represents a condition before the market becomes volatile. If the volatility of the banking sector increases due to a large shock and a counterparty is a risk-averse agent, *A* will move, for example, to *B* reflecting the rising risk (the transaction rate will rise by $\Delta\alpha'$). If the market conditions are extremely uncertain, a bank can be not only risk averse but also uncertainty averse. In this case, *A* will move, for example, to *C* (the trading rate will rise by $\Delta\alpha' + \Delta\beta'$). The premium increases by amplifying the fundamental risks. Although it depends on the definition of reference rates, if we define reference rates as just a tool to measure the degree of "risk free rate + (market and/or credit) risk premium related to the banking sector" excluding the liquidity premium heightened by increased uncertainty ($\Delta\beta'$), we can say the following: *C* is the actual transaction rate, but it contains a number of factors other than the fundamental risk premium; and to see the rate that only reflects the fundamental risk premium, an expert judgment process is necessary to exclude $\Delta\beta'$.

As can be seen currently in the peripheral European sovereign bond markets, yields or interest rates sometimes rise to levels that are well above those that can be justified on the basis of fundamentals (given "bad sentiment" and indescribable unidentified premiums). The economic inefficiency of using interest rates that include such premiums in a number of financial contracts and prices of derivatives should be considered.[13]

5. Conclusion

This study, based on a search model, attempts to draw out the implications for recent discussions about reference rates that originated from the recent Libor manipulation problem, with particular focus on whether the calculation of reference rates should be based solely on actual transaction data and whether the use of expert judgment should

13 For discussions on reference rates and efficiency of the economy as a whole, see for example Sudo (2012).

be allowed to some extent.

The reference rate can be decomposed into elements such as risk-free rate, (credit and/or market) risk premium, and liquidity premium. The theoretical model which combines the search model and robust control method indicated that during periods of market illiquidity (specifically, when market transactions are sporadic), premiums rise while amplifying a jump in reference rates and deviating from fundamentals.

These results imply that it is not desirable to calculate reference rates based solely on actual transaction data in times of crisis, particularly when market liquidity dries up, given the fact that actual reference rates observed in the market include various miscellaneous elements. Although it is important to promote a sound rate-setting process based on the use of actual transaction data, an appropriate and transparent use of expert judgment is also necessary to remove indescribable unidentified premium. It would be difficult, however, for market participants to accurately measure the premium in times of financial crisis, and it is also unreasonable to expect them to make an accurate expert judgment only at moments of financial stress without practice. Therefore, appropriate expert judgment in times of financial crisis requires preparation on a routine basis. It is also necessary to secure a channel through which the transparency enhanced by, for example, improvement of statistical data, reduces the uncertainty associated with model parameters and raises the quality of and confidence in expert judgment. The introduction of robust fallback procedures during periods of market stress might be another way to deal with this issue.

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