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Recent Developments in Measuring the Natural Rate of Interest

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Recent Developments in Measuring the Natural Rate of Interest[*]

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Abstract

The natural rate of interest (r^*) is the real interest rate that is neutral to the economy and prices, and is one of the benchmarks for evaluating the stance of monetary policy. r^* cannot be observed directly and must be estimated based on some assumptions. In this paper, we survey various methods that have been developed for estimating r^*, summarize their characteristics, and apply them to the Japanese economy. We confirm all estimates of r^* showed a downward trend in the long run. However, the estimated results of r^* vary widely, depending on the method used, and current estimates can alter when new data are added to the estimation. Therefore, it is necessary to consider estimation uncertainties when conducting monetary policy.

JEL Classification: C32, E43, E52
Keywords: Natural rate of interest, Equilibrium real interest rate, Equilibrium yield curve, Term-structure

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1. Introduction

The natural rate of interest (r^*) refers to the real interest rate that is neutral to the economy and prices.[1] r^* is an important benchmark indicator for central banks when conducting monetary policy, and various estimation methods have been proposed by academic researchers and central bank economists. However, it has been pointed out that there is a certain amount of uncertainty surrounding the estimates of r^* due to differences in the methods and input data (Brand, Bielecki and Penalver, 2018). For this reason, it is desirable to combine several methods to estimate r^*, with an understanding of the characteristics of each method. In fact, when central banks present estimates of r^*, they often indicate the uncertainty of the estimates by showing the ranges based on several estimation methods.[2,3] In Japan, Kamada (2009), Fujiwara et al. (2016), and Shintani and Miyao (2018) have measured r^* using several estimation methods. Also, since central banks began adopting unconventional monetary policies, research has been gradually accumulating on the "equilibrium yield curve," which extends the concept of r^* to entire yield curve.

In this paper, we survey the estimation methods of r^* that have been proposed so far, organize them by type, and outline their characteristics and points to bear in mind. We also introduce studies that extend the concept of r^* to longer maturity. Then, we estimate r^* in Japan using several representative methods and review the developments.

Our analysis confirms that r^* in Japan showed a gradual downward trend over the long term. However, as has been pointed out in previous studies, there is a certain amount of uncertainty in the estimated values. These characteristics have also been confirmed in recent studies covering the U.S. and the euro area (Benigno et al., 2024).

The structure of this paper is as follows: Section 2 outlines the various methods proposed for estimating r^* and summarizes the characteristics of each type of method. We

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[1] Depending on the economic model from which it is estimated, the natural rate of interest is named differently, such as "the equilibrium real interest rate" or "the neutral interest rate," but this paper makes no distinction between them. Usually, the natural rate of interest is a concept that corresponds to a risk-free, short-term real interest rate.

[2] For example, FRB (2018) describes a range of estimates consisting of seven different estimates of the natural rate of interest. Lane (2022), an ECB Board of Governors speech, describes a range of estimates consisting of eight different natural rates of interest.

[3] Foreign central banks have also frequently disseminated information on the uncertainty of r* estimates. For example, Yellen (2015) mentions the high level of uncertainty in current r* estimates and forecasts. In addition, Bailey (2022), Borio (2021), and Brand, Lisack and Mazelis (2024) state that due to the large uncertainty in r* estimates it should be considered an ex post measure of monetary policy stance rather than a direct guide to monetary policy decision making. In addition to uncertainty in model selection, Brand, Bielecki and Penalver (2018) mention the so-called real-time problem in smoothing methods, where the addition of new data can significantly change past values.

also introduce recent studies that have attempted to estimate equilibrium real interest rates of long maturity, such as the equilibrium yield curve. Section 3 reviews developments in r^* in Japan, obtained from multiple estimation methods, and Section 4 concludes.

2. Estimation methodology of the natural rate of interest

In this section, we categorize and explain various estimation methods for the natural rate of interest (r^*), highlighting their specific characteristics. Additionally, we introduce recent studies that extended the concept of r^* to long-term maturity.

2.1. Types of methodology

The natural rate of interest (r^*), also known as the equilibrium real interest rate or neutral interest rate, has long attracted the attention of economists. The idea that deviations of the real interest rate from r^* lead to fluctuations in prices can be traced back to Wicksell (1898). Later, Woodford (2003) brought renewed attention to this concept by clarifying the relationship between New Keynesian theory and r^*. A representative empirical study of this period is Laubach and Williams (2003). They developed a semi-structural model based on assumptions such as the IS curve and the Phillips curve, and devised a method to estimate the level of r^* consistent with these equations. Their model and the improved models based on their model, which align well with the conduct of monetary policy that considers movements in the output gap, have been widely used by central banks in many countries, even today.[4]

Subsequently, in addition to semi-structural models such as Laubach and Williams (2003), methods have been proposed for estimating r^* based on the term structure of interest rates as a long-run expectation of short-term interest rates (e.g., Kim, Walsh and Wei, 2019) and time-series methods that extract long-run trends in the real interest rate and interpret them as r^* (e.g., Del Negro et al., 2017), expanding the variety of estimation methods.

Obstfeld (2023) categorizes the estimation methods for r^* proposed so far into the following four types (Table 1); namely, (1) time-series models that estimate long-run forecast

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[4] r^* coincides with the level of the potential growth rate in the long run, given some assumptions. Therefore, in the practice of economic analysis, the potential growth rate is often regarded as an approximation of r^*. However, while the potential growth rate is a concept that focuses on the supply side of the economy and attempts to capture its long-term trend, r^* can capture short-term fluctuations in the demand side and may also be subjected to factors specific to financial markets. Therefore, to estimate r^*, it is necessary not only to follow the potential growth rate as a long-term approximation, but also to take into account various short-term factors in the estimation. For a theoretical examination of these points, see, for example, Oda and Muranaga (2003).

Table 1. Type of estimation methods

Type	Idea	Literature
Time series model	Estimating the long-run trend in interest rates, and interpreting this as the natural rate of interest.	▪ Del Negro et al. (2017) ▪ Kiley (2020a)
Term structure model	Using the term structure (yield curve information) to identify the expected path of short-term interest rates, and interpreting this as the natural rate of interest.	▪ Kim, Walsh and Wei (2019) ▪ Bauer and Rudebusch (2020)
Semi-structural model	Assuming some structural equations, such as the IS-curve and the Philips curve, and estimating the interest rate level that makes the output gap zero.	▪ Holston, Laubach and Williams (2023) ▪ Brand and Mazelis (2019)
Structural model	Using micro foundations of the behavior of economic agents to estimate the interest rate level in a perfectly elastic price equilibrium with no nominal rigidities, and taking this to be the natural rate of interest.	▪ Barsky, Justiniano and Melosi (2014) ▪ Okazaki and Sudo (2018)

Notes: The classification is based on Obstfeld (2023).

or trends in real interest rates and regard them as r^* (= time series models), (2) those that estimate long-run expectations of real interest rates in the market using a term structure model of interest rates (= term structure models), (3) semi-structural models, such as Laubach and Williams (2003), which assume a part of the economic structure, and (4) structural models that incorporate microeconomic foundations into the behavior of economic agents. The following sections will summarize the characteristics of each estimation method based on this classification.

2.2. Characteristics of each method

In this section, we will organize the characteristics of each type of estimation method for the natural rate of interest.

2.2.1. Time-series models

Estimation methods using time-series models extract the trend component of the interest rate from real interest rate data. When central banks conduct counter-cyclical monetary policy, exemplified by the Taylor rule (raising the policy rate above r^* during periods of excess demand and rising prices, and lowering it below r^* during periods of excess supply and falling prices), the observed real short-term interest rate fluctuates around r^*. In this case, if the trend in the real interest rate is extracted, it can be interpreted as an estimate of r^*.

Among this type, the simplest approach is to use an HP filter or similar method to extract the trend from the time-series data of real short-term interest rates and regard it as r^* (Kamada, 2009; Fujiwara et al., 2016, etc.). It is the simplest example, but there are more advanced time-series models that also take into account information other than real short-term interest rates, such as price trends. For example, in a representative study by Del Negro et al. (2017) three common trends are extracted from five variables: nominal short-term interest rates, nominal long-term interest rates, market-implied expected future path of the nominal short-term interest rate, inflation rates, and inflation expectations, assuming that these trends are related in a specific manner.[5] Specifically, the following relationships among the trend components of each variable are assumed:

$$\textit{Trend of nominal short-term interest rate} = \bar{\pi}_t + \bar{r}_t \tag{1-1}$$

$$\textit{Trend of nominal long-term interest rate} = \bar{\pi}_t + \bar{r}_t + \overline{tp}_t \tag{1-2}$$

$$\textit{Trend of expected nominal short-term interest rate} = \bar{\pi}_t + \bar{r}_t \tag{1-3}$$

$$\textit{Trend of inflation} = \bar{\pi}_t \tag{1-4}$$

$$\textit{Trend of inflation expectation} = \bar{\pi}_t \tag{1-5}$$

where the barred variables on the right-hand side represent the common trends of each variable. $\bar{\pi}_t$, $\bar{r}_t$ and $\overline{tp}_t$ indicate the trend inflation rate, r^*, and the trend of the term premium, respectively. As is evident from the formulation, in this method, r^* is estimated as the common trend among nominal short-term interest rates, the nominal long-term interest rate, and the expected future path of the nominal short rate. Notably, when extracting the common trend, the assumption is made that "the expectation of a variable and the trend of that variable are identical." Therefore, the expected nominal short-term interest rate in Eq. (1-3) is the forecast of the nominal short-term interest rate in Eq. (1-1), and the trends of these two variables are represented by the same equation. This relationship also applies to the inflation rate in Eq. (1-4) and inflation expectations in Eq. (1-5) (see Appendix A for

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[5] A related method is Johannsen and Mertens (2021). This method is similar to Del Negro et al. (2017) in that it extracts common trends in interest rates for multiple maturities, but by using shadow rates in the estimation, the model conceptually removes the effect of the effective lower bound on nominal interest rates. In addition, Kiley (2020a) estimates global r^* by extracting common trends from short-term and long-term interest rates for 13 advanced economies and then adjusting for specific factors to estimate r^* for each country; Del Negro et al. (2019) extracts common trends in interest rates in developed countries to account for global factors.

more details).

2.2.2. Term structure models

Estimation methods using term structure models utilize not only the real interest rate of a specific maturity, but also the information from the term structure of interest rates (the shape of the yield curve). This information is used to obtain the short-term implied forward rate for the future – the market-implied expected future path of the real interest rate– where sufficient time has passed from the present, and the effects of short-run economic shocks have subsided. Then, this is regarded as r^*. In this process, since r^* generally corresponds to the concept of a risk-free interest rate, it is necessary to identify and exclude the effects of various premiums corresponding to each maturity in the estimation. For example, Kim, Walsh and Wei (2019) use a term structure model based on the no-arbitrage condition proposed by D'Amico, Kim and Wei (2018, hereafter DKW) to estimate such premiums and attempt to extract r^* in the United States.

The DKW model was originally designed to decompose the premium contained in the yield on U.S. treasury inflation-protection securities (TIPS). Using the DKW model, the nominal interest rate ($R_{\tau,t}$), the yield on price indexed bonds ($TIPS_{\tau,t}$) and the real interest rate ($r_{\tau,t}$) at time t and maturity τ, can be decomposed as follows.

$$R_{\tau,t} = r_{\tau,t} + \text{inflation expectation} + \text{inflation risk premium} \tag{2-1}$$

$$TIPS_{\tau,t} = r_{\tau,t} + \text{TIPS liquidity premium} \tag{2-2}$$

$$r_{\tau,t} = \underline{\text{expectation of real short-term rate }} + \text{term premium} \tag{2-3}$$

Therefore, in the DKW model, the short-term implied forward rate, which is the expected component of the real short-term interest rate, can be extracted after taking into account trends in the various premiums (liquidity premium, inflation risk premium, and term premium) included in the yields on nominal government bonds and inflation-protected securities. The real short-term interest rate forecast for 5 to 10 years ahead can be interpreted as an estimate of the equilibrium level of the short-term interest rate, or r^*, since the effects of short-lived economic shocks are likely to have largely disappeared.

2.2.3. Semi-structural models

In the estimation method using the semi-structural model, r^* is estimated as the "economy-neutral interest rate with zero output gap" based on the explicit assumption of structural equations representing relationships such as the IS curve and the Phillips curve with respect to each economic variable.

Although there are many variations of semi-structural models, depending on their assumptions about economic structure, the three core structural equations in most models, such as those of Laubach and Williams (2003), are as follows:

$$y_t - y_t^* = \beta(r_t - r_t^*) \tag{3-1}$$

$$\pi_t = \pi_{t-1} + \theta(y_t - y_t^*) \tag{3-2}$$

$$r_t^* = \sigma g_t^* + z_t \tag{3-3}$$

where y_t is the output, y_t^* is the potential output, r_t is the real interest rate, π_t is the inflation rate, g_t^* is the trend in potential output, and z_t is other factors that affect the level of r_t^*, such as time preference rate. Eq. (3-1) is the IS curve that represents the relationship between the output gap $(y_t - y_t^*)$ and the real interest rate gap $(r_t - r_t^*)$. It assumes $\beta < 0$ based on standard economic theory and depicts the relationship that the more the real interest rate gap expands in the negative direction, the more the positive range of the output gap expands. Eq. (3-2) is a Phillips curve that expresses the relationship between the output gap and the inflation rate, and depicts the relationship that a positive expansion of the output gap increases the inflation rate under $\theta > 0$. Eq. (3-3) is an equation that defines the relationship between r_t^* and the potential growth rate, and although the two basically move in the same direction, the existence of z_t allows them to diverge to a certain degree. Based on these structural equations, r_t^* and potential output are estimated simultaneously using the Kalman filter. According to the relationship in eq. (3-1), r^* is estimated as the level of the real short-term interest rate at which the interest rate gap is zero, thus the output gap is also zero.

Since the work by Laubach and Williams (2003), various improvements and proposals have been made to the formulation of the structural equations. For example, Kiley (2020b) added long-term inflation expectations into the Phillips curve, and since the accuracy of estimating the output gap improves when information on the unemployment rate is used, he added Okun's law, which expresses the relationship between output and the unemployment rate, to the structural equation. In addition, Holston, Laubach and Williams (2023, hereafter HLW) explicitly incorporated supply shocks such as public health measures into their model to deal with extreme movements in economic variables associated with the COVID-19 pandemic, and made other improvements such as allowing time-varying variance (see Appendix B for details).

2.2.4. Structural models

Although semi-structural models assume a certain economic structure, such as the IS curve and the Phillips curve, they are not general equilibrium models based on the dynamic optimization behavior of economic agents such as households and firms. To take this into account, there are studies that use structural models that provide a microeconomic foundation for the behavior of economic agents. In the case of structural models, r^* is defined and estimated as the real short-term interest rate that would exist if prices and nominal wages were perfectly elastic (Woodford, 2003). The structural models used to estimate r^* can be broadly classified into the New Keynesian Dynamic Stochastic General Equilibrium Model (DSGE model) and the Overlapping Generations Model (OG model).

Kamada (2009) summarizes the characteristics of estimation using a structural model in the following three points: First, since the model is based on economic theory, it is easy to explain the movement of endogenous variables within a basic econometric framework. Second, depending on the degree of detail in the model, it should be possible to identify the deviation of the data from the econometric model, i.e., what factors caused the shocks. Third, since the model is specified by parameters that do not alter with policy changes (deep parameters), it is not subject to the Lucas critique when conducting policy simulations.

Okazaki and Sudo (2018) is one of the previous studies applying a DSGE model to the Japanese economy.[6] By introducing stochastic fluctuations in the working-age population and other factors into the standard New Keynesian model, they have constructed a model that enables comparison of the relative importance of each factor that affects r^*.[7]

Figure 1 shows the decomposition of the change of r^* in Japan using the DSGE model. It can be seen that a significant portion of the variation in r^* from the 1990s to the mid-2010s can be explained by productivity factors (changes in neutral technology). In addition, demographic factors also contributed to the trend decline in r^*, but from a quantitative perspective, their contribution was limited.[8] In this way, the ability to perform factor decomposition is a characteristic and advantage of the structural model.

The OG model explicitly models multiple generations with different ages, focusing on age-specific heterogeneity of households in terms of savings, labor supply, etc., and has implications for the relationship between demographic changes and macroeconomic

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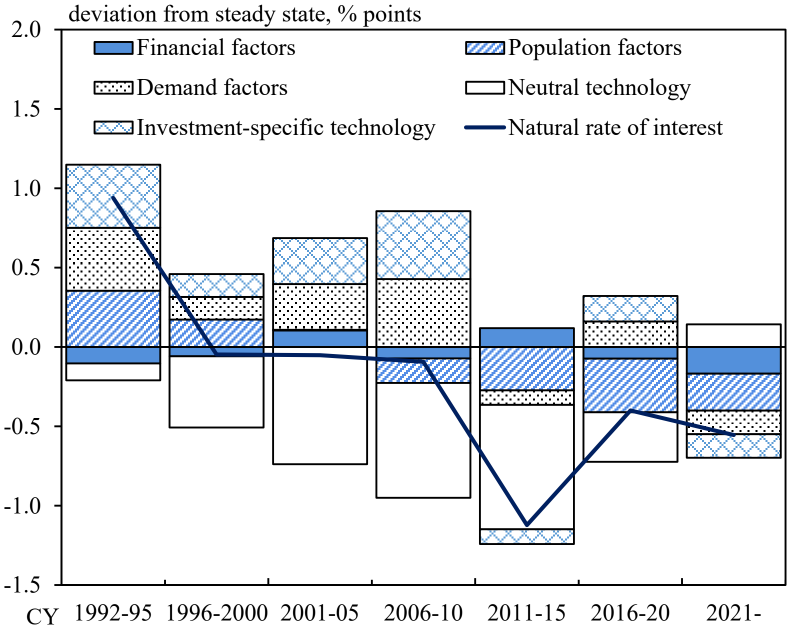
[6] For other previous studies, see Kiley (2013), Barsky, Justiniano and Melosi (2014), Christiano, Motto and Rostagno (2014), and Guerrieri and Iacoviello (2017). Hirose and Sunagawa (2023) use U.S. data to estimate a nonlinear DSGE model that takes into account an effective lower bound on the nominal interest rate. For other studies using Japanese data, besides Okazaki and Sudo (2018), see Iiboshi, Shintani and Ueda (2022), for example.

[7] Specifically, the study takes into account factors such as neutral technology (TFP), degree of functioning of the financial intermediation, demographics, investment-specific technology, and changes in the subjective discount rate.

[8] It is worth noting that the demographic factors in this model capture the impact of unanticipated shocks to demographics and do not include the impact of a priori anticipated demographic changes.

Figure1: Decomposition of the fluctuation of r* based on Okazaki and Sudo (2018)



Notes: Financial factors, Population factors, Demand factors, Neutral technology, and Investment-specific technology show the contributions to the natural rate of interest of, respectively, shocks to net assets of enterprises and banks, working-age population, demand structure for each goods, TFP, and productivity for production of capital goods,.

Source: Authors' estimation; Bank of Japan; Ministry of Health, Labour and Welfare; Ministry of Internal Affairs and Communications, and Bloomberg, etc.

variables. The model allows us to more explicitly describe the effect of low fertility and aging on r^* through capital deepening (increase in capital relative to labor), as pointed out in studies by Carvalho, Ferrero and Nechio (2016) and Eggertson, Mehrotra and Robbins (2019). In the case of Japan, it is significant to evaluate the role of demographic change, given that the falling birthrate and aging population are expected to continue to progress in the future.

Based on this awareness of the problem, Sudo and Takizuka (2020) and Katagiri et al. (2024) conducted OG model simulations for Japan. Katagiri et al. (2024) argue that although r^* in Japan, has been pushed down by more than -100 bps due to declining birthrates and increasing longevity since 1980, the demographic changes expected in the future are not likely to have the effect of causing r^* to deviate significantly from the current level compared to the demographic changes that have already occurred. One of the features of the OG model is that it can simulate long-run changes in r^* over the next 50 years, taking into account the relatively easy-to-predict impact of demographic changes among the factors that affect r^*.

2.3. Extension to the long-term natural rate

Since the global financial crisis of 2008, many central banks have introduced unconventional monetary policies such as large-scale asset purchases and forward guidance. These policies are unique in that they work not only on short-term policy rates but also on the entire yield curve. For example, the Bank of Japan implemented yield curve control (YCC) from 2016 to 2024, which included policy targets for not only short-term but also long-term interest rates. It has been pointed out that in such a framework there is a need to establish criteria for assessing the degree of monetary accommodation, including the effects of unconventional monetary policy (Kuroda, 2017). With this in mind, academic research on the "equilibrium yield curve" has been gradually accumulating (Brzoza-Brzezina and Kotłowski, 2014; Dufrénot, Rhouzlane and Vaccarro-Grange, 2022, etc.), with the extension of the concept of the natural rate of interest, which had been limited to short-term maturity, to long-term real interest rates and the entire yield curve.

The equilibrium yield curve is an extension to all maturities of the conventional idea of the single-maturity natural rate of interest (Imakubo, Kojima, and Nakajima, 2015). If the actual real yield curve matches the equilibrium yield curve, the output gap will converge to zero. In addition to this study, Roberts (2018) attempts to estimate r^* for the longer-term, based on the Laubach and Williams (2003) model. He uses the 10-year bond rate in the IS curve instead of the short-term interest rate. Although this is a simple framework, it is noteworthy in the sense that the importance of the long-term equilibrium rate has been pointed out in the United States by Roberts (2018).

In light of these research, this section will focus on two related studies of equilibrium yield curves: Hatayama and Iwasaki (2024), which takes a time-series approach, and Imakubo, Kojima, and Nakajima (2015), which uses a semi-structural approach.

2.3.1. Time-series approaches

Hatayama and Iwasaki (2024) attempt to estimate the equilibrium yield curve in Japan using the time series method proposed by Goy and Iwasaki (2024). Goy and Iwasaki (2024) estimate the equilibrium yield curve by combining the Nelson-Siegel model[9] with a macroeconomic model. Specifically, they assume the following relationships for the trends of real interest rates for each maturity, nominal interest rates for each maturity, inflation rates, and inflation expectations, and extract four common trends.

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[9] This is a yield curve model developed by Nelson and Siegel (1987) that decomposes the shape of the yield curve into three factors: level, slope, and curvature.

$$\textit{Trend of real interest rate} \;=\; \bar{r}_t + (\theta_S(\tau,\lambda_r)-1)\bar{s}_t \tag{4-1}$$

$$\textit{Trend of nominal interest rate} \;=\; \bar{r}_t + (\theta_S(\tau,\lambda_r)-1)\bar{s}_t + \bar{\pi}_t + (\theta_S(\tau,\lambda_\pi)-1)\bar{s}_t^{\pi} \tag{4-2}$$

$$\textit{Trend of inflation rate} \;=\; \bar{\pi}_t \tag{4-3}$$

$$\textit{Trend of inflation expectation} \;=\; \bar{\pi}_t \tag{4-4}$$

where $\bar{r}_t$ is the trend of the real short-term interest rate, $(\theta_S(\tau,\lambda_r)-1)\bar{s}_t$ is the trend of the term premium for maturity τ, $\bar{\pi}_t$ is the common trend of inflation rate and inflation expectations, and $(\theta_S(\tau,\lambda_\pi)-1)$ is the slope trend of the inflation expectations curve for maturity τ. Note that $(\theta_S(\tau,\lambda_r)-1)$ and $(\theta_S(\tau,\lambda_\pi)-1)$ are factors that increase monotonically with maturity τ. Using this model, we can obtain the equilibrium value of the real short-term interest rate and the equilibrium value of the term premium for each maturity, which can be summed to obtain the equilibrium real interest rate corresponding to each maturity. For a more detailed explanation of the estimation method and results, see Hatayama and Iwasaki (2024).

In addition, the Del Negro et al. (2017) model we referred in section 2.2.1 can be used to estimate the short-term r^* and the trend of the term premium corresponding to a specific maturity (10-year). Therefore, an equation such as the following:

Long-term equilibrium real interest rate = Short-term r^*+ trend of the term premium

can be used to estimate the level of r^*corresponding to the 10-year real interest rate, but not the entire yield curve.[10]

2.3.2. Semi-structural model approaches

Imakubo, Kojima, and Nakajima (2015) and Nakajima et al. (2023) estimate the equilibrium yield curve by adopting semi-structural models. The models used in these studies are based on a semi-structural model in Laubach and Williams (2003), in which the IS curve is modified to describe the relationship between the yield curve gap (the gap between the actual real yield curve and the equilibrium yield curve) and the output gap. Specifically, the real yield curve is first decomposed into three factors using the Nelson-Siegel model. Then, the relationship between the gap from the equilibrium value of each factor and the output gap is formulated as follows.

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[10] In a related study, Kopp and Williams (2018) attempt to extract the trend component of the real long-term interest rate, taking into account the relationship between the yield curve and macroeconomic variables such as the unemployment rate.

$$y_{\mathrm{t}} - y_{\mathrm{t}}^{*} = \beta_L(L_t - L_t^{*}) + \beta_S(S_t - S_t^{*}) + \beta_C(C_t - C_t^{*}) \tag{5-1}$$

where L_t, S_t, and C_t in the equations denote the level, slope, and curvature factors of the yield curve as estimated by the Nelson-Siegel model. Comparing the IS curve of the Laubach and Williams model in eq. (3-1), one can see that the IS curve is changed in that the three factors from the yield curve decomposition affect the output gap in eq. (5-1). Under this assumption, the financial environment is neutral when all three factors are equal to their equilibrium values (L_t^{*}, S_t^{*}, C_t^{*}). Such a formulation allows us to estimate the shape of the economy-neutral yield curve.[11]

2.4. Characteristics of the estimated r* from each type of estimation methodology

In this section, we will discuss the characteristics and caveats of the four types of methodology introduced thus far.

The characteristics of the estimates by type are particularly influenced by the strength of their assumptions regarding the economic structure that affects the movement of r^{*}.

Time-series models and term structure models generally do not make strong assumptions on the economic structure and instead focus on observed interest rate information, emphasizing a data-driven approach. Because of this, it is relatively easy to avoid issues arising from the formulation of the model, such as how to incorporate economic variables into the model and the possibility that the parameters of structural equations may change over time. On the other hand, it should be noted that the model tends to reflect actual observed real interest rates due to the emphasis on real interest rate information in the estimation.[12] In addition, although the term structure model estimates r^{*} by removing various premiums from the observed data, it is hard to remove the premiums completely, thus, the estimates can be more easily influenced by observed data.

On the other hand, the semi-structural model, which assumes structural equations such as the IS curve and the Phillips curve and considers r^{*} as the level of interest rates

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[11] Dufrénot, Rhouzlane and Vaccaro-Grange (2022) extends Imakubo, Kojima, and Nakajima (2015). They estimate their equilibrium yield curve model by adding factors such as the impact of financial factors and fiscal spending to the model of Imakubo, Kojima and Nakajima (2015). In addition, Brand, Goy and Lemke (2021) also attempt to estimate an equilibrium yield curve model based on the Laubach and Williams (2003) model by improving the model to allow for the term structure of interest rates, but unlike Imakubo, Kojima and Nakajima (2015), the assumption is made that the slope and curvature components of the yield curve are constant.

[12] In particular, the estimation based on a time-series model implicitly assumes that the actual real interest rate circulates around the natural rate of interest in line with economic fluctuations and other factors. Therefore, it should be noted that such restrictions may affect the estimated values when fluctuations in the real interest rate are restricted by the effective lower bound.

consistent with them, has the advantage that r^* fluctuates in the direction assumed by economic theory, making it easier to interpret the factors of fluctuation in relation to fluctuations in the real economy. On the other hand, the estimation results strongly depend on the assumptions of the model. For example, Nakajima et al. (2023) compared a formulation in which r^* depends on the <u>level</u> of potential GDP with a formulation in which r^* depends on the <u>rate of change</u> of potential GDP when estimating an equilibrium yield curve model, and found that the estimated value of r^* can vary correspondingly. Even in the structural model, there can be many variations in the economic structure assumed, and the estimation results are more dependent on the model setting.[13]

Furthermore, Taylor and Wieland (2016) point out that for the simple IS curve assumed in the semi-structural model, the estimated r^* may be affected by missing variables if there are other important factors that determine the output gap besides the short-term interest rate gap. That is, if the IS curve shown in eq. (3-1) is in fact,

$$\mathrm{y_t} - \mathrm{y_t^*} = \beta(r_t - r_t^*) \ - a x_t^* \tag{6-1}$$

then, the β estimated by eq. (3-1) lack consistency, and r^*calculated on those assumptions can also be affected by errors in formulation.[14]

From the above discussion, the various approaches have their own advantages and disadvantages, and there is no clear superiority or inferiority. Therefore, when estimating r^*, it is desirable not to rely on a single model, but to use multiple estimation methods and cross-check them to reduce as much as possible any uncertainty arising from model selection. When doing so, instead of simply increasing the number of models used indiscriminately, one should be mindful of the types and characteristics of estimation methods and strive for a balanced model selection.

3. Measuring r* in Japan

3.1. Our methodology

In this section, we estimate r^* using various estimation methods introduced in the previous sections and examine the r^* in Japan.

The models were selected in consideration of the balance among the types introduced

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[13] Shintani and Miyao (2018) estimated r^* for Japan using four different DSGE models, linear/non-linear and with/without consideration of the zero lower bound. They report that $r^*$is estimated higher when a linear model is used, and that the consideration of the zero lower bound had no qualitative impact on the estimated value.

[14] Taylor and Wieland (2016) provide examples of government regulatory and fiscal policies, such as tax changes on capital investment, as potential missing variables.

in the previous section. As a time-series method, the Del Negro et al. (2017) model is applied to Japan,[15] and the Goy and Iwasaki (2024) model obtained in Hatayama and Iwasaki (2024) is also reported. As a semi-structural model, we apply the HLW (2023),[16] which is a modified version of Laubach and Wiliams (2003), as well as extended estimates of two equilibrium yield curve models by Imakubo, Kojima, and Nakajima (2015) and Nakajima et al. (2023).[17] As a structural model, we extended the DSGE model by Okazaki and Sudo (2018), which explicitly takes into account population changes and the role of the financial sector. These allow us to confirm the trends by a total of six methods for the short-term $r^*$and four methods for the long-term equilibrium interest rate (Table 2).

Table 2. Estimation methodology

Model	Type	Overview and Features
Del Negro et al. (2017)	Time series	・Extracting trends in real interest rates from short- and long-term nominal interest rates, inflation rates, inflation forecasts, etc., and considering them as r*. ・Estimates tend to be affected by actual real interest rates.
Goy and Iwasaki (2024)	Time series	・Estimating r* as a common trend based on real interest rates for each maturity. In doing so, information on the real economy, such as the output gap, is also taken into account. ・Estimates tend to be affected by actual real interest rates.
Holston, Laubach and Williams [HLW] (2023)	Semi-structural	・Assuming the IS curve, Phillips curve, and other structural equations, and estimating the interest rate level that makes the output gap zero in each period. ・Estimates tend to be affected by real GDP, which is used as an input.
Imakubo, Kojima and Nakajima (2015)	Semi-structural	・Extension of the HLW model. Not only is r* for a single maturity, but the economy-neutral shape of the yield curve (equilibrium yield curve) is also estimated. ・Estimates tend to be affected by potential GDP, which is used as an input.
Nakajima et al. (2023)	Semi-structural	
Okazaki and Sudo (2018)	Structural	・Estimating r* based on a DSGE model of the standard setup plus factors such as financial market imperfections and demographics.

Notes: Imakubo, Kojima, and Nakajima (2015) estimate the equilibrium yield curve which depends on the level of potential GDP, and Nakajima et al. (2023) estimate that which depends on the rate of change of potential GDP.

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[15] See Appendix A for details.

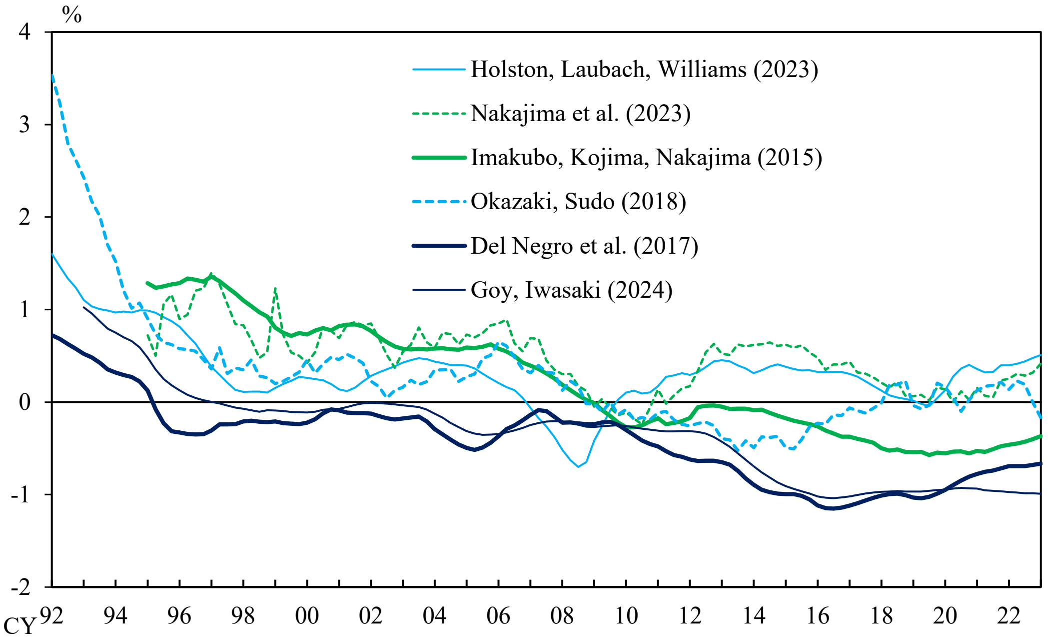
[16] See Appendix B for details.

[17] See Appendix C for details.

3.2. Natural rate of interest in Japan

In this section, we review r^* in Japan as estimated by several methods (Figure 2).

Figure 2. r* in Japan



Notes: Del Negro et al. (2017), HLW (2023), Imakubo, Kojima, and Nakajima (2015), Nakajima et al. (2023), and Okazaki and Sudo (2018) were estimated by the authors; Goy and Iwasaki (2024) was estimated by Hatayama and Iwasaki (2024).

Sources: Bank of Japan; Ministry of Finance; Ministry of Health, Labour and Welfare; Ministry of Internal Affairs and Communications; Cabinet Office; Bloomberg; Consensus Economics "Consensus Forecast;" Google Mobility Index.

First, a long-term declining trend can be observed for all methods. This result is consistent with the declining trend in global r^* confirmed in previous studies.[18] In detail, some estimates fell below zero for the first time around 1995, followed by a gradual downward trend, with almost all estimates taking negative values around 2010.[19] In subsequent years, many estimates have remained generally unchanged.

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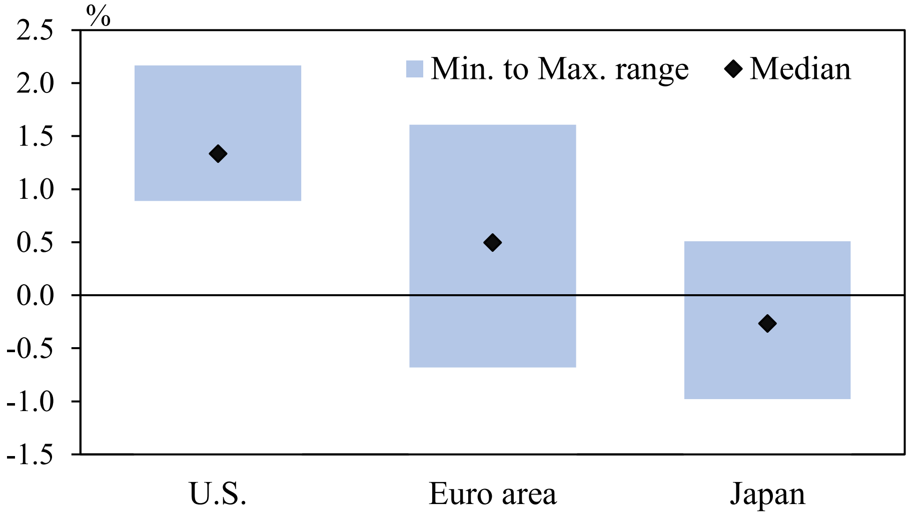
[18] The decline in r^* can be attributed to (i) lower potential growth (Summers, 2014; Cesa-Bianchi, Harrison and Sajedi, 2022), (ii) demographic factors (Carvalho, Ferrero and Fernande, 2016; Auclert et al. 2021), (iii) increased demand for safe assets (Del Negro et al., 2017), (iv) widening inequality (Mian, Straub and Sufi, 2021), and (v) global factors (Rachel and Smith, 2017).

[19] There is a debate as to whether r^*can be negative on a permanent basis, in relation to the "secular stagnation" raised by Summers (2014). Although the factors that led to a negative r^* include the credit contraction after the financial crisis (Eggertsson, Mehrotra and Robbins, 2019) and the existence of convenience yield (Del Negro et al., 2017), both of these arguments emphasize financial factors. Bernanke (2015) acknowledged the existence of these factors, but argued that it is doubtful that r^* can be negative over a long period, since a negative real interest rate would make most investment projects profitable. Summers (2015) countered that negative real interest rates are a theoretically obscure but practically

Figure 3. Range of the estimated value of r* (U.S., euro area, and Japan)



Notes: The results for the U.S. and Europe are based on Benigno et al. (2024) and show the maximum to minimum range of estimates from 2023/3Q to 2024/4Q. See Benigno et al. (2024) for details of the estimation methodology used. Results for Japan are calculated by the authors using 2023/1Q estimates.

Sources: Bank of Japan; Ministry of Finance; Ministry of Health, Labour and Welfare; Ministry of Internal Affairs and Communications; Cabinet Office; Bloomberg; Consensus Economics "Consensus Forecast;" Google Mobility Index.

It was also confirmed once again that there is a reasonable range in the level of r^* estimates depending on the method used. However, this is not a unique phenomenon to Japan. Benigno et al. (2024) estimated recent r^* values in the U.S. and the euro area using multiple methods and showed the range of r^* values as in this paper. Figure 3 compares the U.S. and European results with those of this paper. From this, it can be seen that the uncertainty of r^* in Japan is about the same as that estimated for the U.S. and Europe.

Some estimation methods also allow estimation of the level of the equilibrium real interest rate for the long-term maturity as well as short-term r^*. The results of the estimation of the equilibrium real interest rate for the long-term maturity (10-year) are shown in Figure 4.

This confirms the long-term declining trend as well as the short-term r^*. Similarly, there is significant uncertainty in the estimates.

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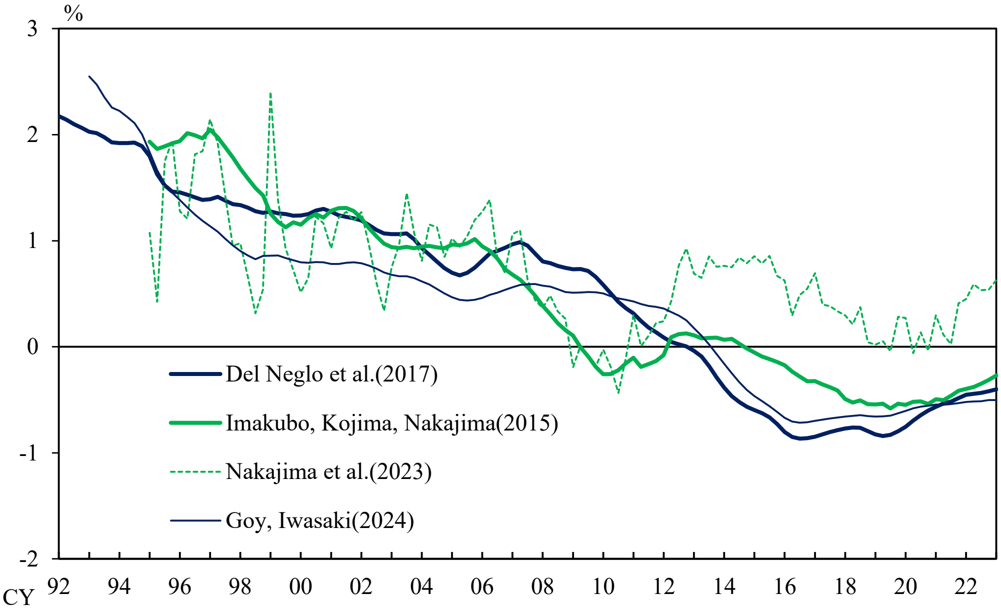
observed phenomenon, citing the empirical finding in Hamilton et al. (2015) that real interest rates were likely negative in the United States for a long time in the 20th century.

Figure 4. Long-term equiribrium interest rates



Notes: Del Negro et al. (2017), Imakubo, Kojima, and Nakajima (2015), and Nakajima et al. (2023) are estimated by the authors; Goy and Iwasaki (2024) model are estimates by Hatayama and Iwasaki (2024).

Sources: Bank of Japan; Ministry of Finance; Ministry of Health, Labour and Welfare; Ministry of Internal Affairs and Communications; Cabinet Office; Bloomberg; Consensus Economics "Consensus Forecast."

4. Concluding Remarks

In this paper, we surveyed the r^* estimation methods that have been proposed so far, categorized them, and identified the characteristics of each type. Then, in order to deal with the estimation uncertainty, we estimated r^* in Japan using several methods and confirmed the trend of r^*. The estimation results show that r^* in Japan has been on a gradual downtrend over a long period. The decline in r^* increased the likelihood that the nominal interest rate would touch the effective lower bound constraint, which may have limited the monetary policy responses through short-term interest rate control. However, there are large differences in the estimated values depending on the estimation method, and as has been pointed out in previous studies, there is a certain amount of uncertainty in the estimates.

One point to bear in mind when referring to the estimates presented in this paper is the uncertainty inherent in the estimation of each individual method. In other words, while the combination of multiple estimation methods in this paper addressed the uncertainty in model selection to a certain extent, the uncertainty in each estimation method remains. Taking this into account, the range of estimation results is wider. Furthermore, it has been pointed out that there is a so-called "real-time problem," in particular with regard to the recent estimates, in that past values may change significantly when new observed data are added for estimation. Thus, the possibility cannot be ruled out that the current estimates may differ from those in the past. It should be noted that it is not easy to pinpoint the current level of r^* due to these factors.

Future work includes improving each empirical model in order to reduce the uncertainty of individual estimation methods. In addition to this, with regard to long-term r^*, although there is an accumulation of empirical research, the theoretical work about the determinants and dynamics of long-term r^* is far from satisfactory. In addition to improving the empirical models, we hope to deepen our theoretical understanding.

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Appendix A. Summary of the method of Del Negro et al. (2017)

Del Negro et al. (2017) extracts r^* as one of the common trends from multiple economic variables, including interest rates and the inflation rate. This method has the characteristic of using more information than the simple method, such as applying an HP filter to extract a trend in the short-term real interest rate.

For the specific estimation procedure, we follow Del Negro et al. (2017) and consider the following state space representation. First, the observation equations are as follows.

$$R_t^{short} = \bar{\pi}_t + \bar{r}_t + \tilde{R}_t^{short} \tag{A1-1}$$

$$R_t^{long} = \bar{\pi}_t + \bar{r}_t + \overline{tp}_t + \tilde{R}_t^{long} \tag{A1-2}$$

$$R_t^{e,short} = \bar{\pi}_t + \bar{r}_t + \tilde{R}_t^{e,short} \tag{A1-3}$$

$$\pi_t = \bar{\pi}_t + \tilde{\pi}_t \tag{A1-4}$$

$$\pi_t^e = \bar{\pi}_t + \mu^\pi + \tilde{\pi}_t^e \tag{A1-5}$$

where the variables R_t^{short}, R_t^{long}, $R_t^{e,short}$, π_t, and π_t^e in the left-hand side of each equation represent the nominal short-term interest rate, nominal long-term interest rate, expected future path of the nominal short rate, inflation rate, and expected inflation rate, respectively. The barred variables on the right-hand side represent the trend terms. $\bar{\pi}_t, \bar{r}_t$ and $\overline{tp}_t$ represent the time-varying trend of the inflation rate, real short-term interest rate, and term premium. In addition, reflecting the fact that the actual inflation rate has been below the expected inflation rate for a long period in Japan (on average, the expected inflation rate has been above the inflation rate by about 1% over the estimation period, discussed below), we added a bias term μ^π to capture the assumed trend level differences. Finally, the tilde terms on the right-hand side represent the cyclical components of each variable.

In extracting common trends from multiple economic variables, this estimation makes the assumption that the forecast for a variable coincides with the trend for that variable. For example, the expected future path of the nominal short rate on the left side of Eq. (A1-3) is the expected value of the nominal short-term interest rate on the left side of Eq. (A1-1). Thus, both variables have a common trend $\bar{\pi}_t + \bar{r}_t$, and each cycle term $\tilde{R}_t^{short}, \tilde{R}_t^{e,short}$ captures the short-term difference.

This model can be represented in state space by combining the observation equations described earlier with the transition equations that model the variation of the state variables. For the state variables, the time-varying trend term $\bar{y}_t \equiv (\bar{\pi}_t, \bar{r}_t, \overline{tp}_t)'$ and the cycle term $\tilde{y}_t \equiv (\tilde{R}_t^{short}, \tilde{R}_t^{long}, \tilde{R}_t^{e,short}, \tilde{\pi}_t, \tilde{\pi}_t^e)'$ are aggregated into q×1 and n×1 vectors ($q = 3,\ n = 5$). The transition equations are given below under the assumptions that (1) the trend term follows a unit root process and (2) the cycle term follows a VAR process.

$$\bar{y}_t = \bar{y}_{t-1} + e_t \tag{A1-6}$$

$$\tilde{y}_t = \Phi(L)\tilde{y}_{t-1} + \epsilon_t \tag{A1-7}$$

where L represents the lag operator, $\Phi(L) \equiv \sum_{s=1}^{p} \Phi_s L^{s-1}$ and Φ_s is an n×n matrix. The shocks pertaining to the trend-cycle terms are $(\mathrm{q}+\mathrm{n}) \times 1$ vectors and follow the i.i.d. process in the following form.

$$\begin{pmatrix} e_t \\ \epsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0_q \\ 0_n \end{pmatrix}, \begin{pmatrix} \Sigma_e & 0 \\ 0 & \Sigma_\epsilon \end{pmatrix}\right) \tag{A1-8}$$

where $N(\cdot,\cdot)$ represents the multivariate normal distribution, and the variance-covariance matrix Σ_e and Σ_ϵ are $\mathrm{q} \times \mathrm{q}$, $\mathrm{n} \times \mathrm{n}$ positive definite matrices.

Based on such a state space representation, we estimate r^* based on a Bayesian estimation method. The parameters to be estimated are the bias term μ^π, the VAR coefficients $\varphi \equiv vec(\Phi)$ for the cycle term, and the variance-covariance matrices Σ_e and Σ_ϵ for the shock terms in the transition equation. μ^π is obtained by the Metropolis-Hasting algorithm and the other parameters are obtained by the Gibbs sampler. The state variables in each period $(\bar{y}_t, \tilde{y}_t)$ are sampled by the method proposed by Durbin and Koopman (2002).

The prior distribution is also given below according to Del Negro et al. (2017).

$$p(\varphi|\Sigma_\epsilon) = N\left(vec\left(\underline{\varphi}\right), \Sigma_\epsilon \otimes \underline{\Omega}\right) I(\varphi) \tag{A1-9}$$

$$p(\Sigma_e) = IW\big(\kappa_e, (\kappa_e + q + 1)\underline{\Sigma}_e\big) \tag{A1-10}$$

$$p(\Sigma_\epsilon) = IW\big(\kappa_\epsilon, (\kappa_\epsilon + n + 1)\underline{\Sigma}_\epsilon\big) \tag{A1-11}$$

$$p(\mu^\pi) = N\left(\underline{\mu}^\pi, \underline{\sigma}^2_{\mu^\pi}\right) \tag{A1-12}$$

where $IW\big(\kappa, (\kappa + n + 1)\underline{\Sigma}\big)$ is an inverse-Wishart distribution with mode $\underline{\Sigma}$ and degrees of freedom κ. $I(\varphi)$ is an indicator function such that it takes 1 if the VAR process indicated by the parameter φ satisfies the stationarity condition.

In addition, for each state variable $\bar{y}_0, \tilde{y}_{0:-p+1} \equiv \left(\tilde{y}'_0, \tilde{y}'_{-1}, \ldots, \tilde{y}'_{-p+1}\right)'$, the initial values follow the following distribution.

$$\bar{y}_0 \quad \sim \quad N\left(\underline{y}_0, \underline{V}_0\right) \tag{A1-13}$$

$$\tilde{y}_{0:-p+1} \quad \sim \quad N\left(0, V(\Phi, \Sigma_\epsilon)\right) \tag{A1-14}$$

where $V(\Phi, \Sigma_\epsilon)$ represents the variance of the stationary distribution implied by Eq. (A1-7).

Next, we will outline the data used in the estimation. First, the estimation period is from 1992/Q1 to 2023/Q1. It covers the 1990s, when a decline in the natural rate of interest is said to have begun in Japan, as pointed out in previous studies such as Fujiwara et al. (2016). Next, regarding the specific data used, we use the call rate as the short-term nominal interest rate R_t^{short} and the 10-year JGB rate as the nominal long-term interest rate R_t^{long}. For most of the periods covered by the estimation, the call rate has been subject to the effective lower bound, which may also distort the trend cycle estimation. Because of this issue, as in Del Negro et al. (2017), the call rate is not used in the estimation for the periods when the Bank of Japan adopted a zero or negative interest rate policy (1999/Q2 to 2006/Q2 and 2010/Q4-). Next, the expected future path of the nominal short rate $R_t^{e,short}$, is the expected short-term interest rate component corresponding to the 10-year zero coupon yield based on Kaihatsu et al. (2024). Normally, the expected future path of the nominal short rate is estimated using some kind of term structure model of interest rates or macroeconomic model. In this paper, we use the results estimated in Kaihatsu et al. (2024) using the term structure model proposed by Imakubo and Nakajima (2015). Finally, for the inflation rate π_t, we use the year-on-year change in the consumer price index (less fresh food, consumption tax adjusted), and for the expected inflation rate π_t^e, we use the "Consensus Forecast" 6 to 10 years ahead (consumption tax adjusted). We also use values consistent with Del Negro et al. (2017) for parameters such as prior distributions. First, with respect to the initial value of the trend term $N\left(\underline{y}_0, \underline{V}_0\right)$, $\underline{y}_0 = (\underline{\pi}_0, \underline{r}_0)'$ is set to $\left(\underline{\pi}_0, \underline{r}_0, \underline{tp}_0\right)' = (1, 1, 0.5)'$ based on previous studies for the natural rate of interest in Japan, and the variance-covariance matrix for the initial value of the trend term $\underline{V}_0$ is a diagonal matrix with a diagonal component of 0.1. For the prior distribution $p(\Sigma_e)$ on the variance of the trend term ($\underline{\Sigma}_e$) is $(\bar{\pi}_t, \bar{r}_t, \overline{tp}_t)$, where the corresponding diagonal components are (2/400, 2/400, 1/1600)[20] and $\kappa_e = 100$. Next, the order of the VAR process that the cycle term follows is set to p=5 as in Del Negro et al. (2017), and the prior distribution $p(\varphi|\Sigma_\epsilon)$ for the coefficients of the VAR also follows the standard Minnesota Prior, except that the self-lag coefficient is set to 0. The prior distribution

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[20] The parameters are set to satisfy the following conditions: (i) the variance of $\bar{r}_t$ is consistent with the variance of the estimates of the natural rate of interest for Japan presented in previous studies, (ii)the sum of the variance of $\bar{r}_t$ and$\overline{tp}_t$ is consistent with the variance of the Bank of Japan's estimate of the potential growth rate.

for the variance of the cycle term $p(\Sigma_\epsilon)$ is a diagonal matrix, and the components corresponding $(\tilde{R}_t^{short}, \tilde{R}_t^{long}, \tilde{R}_t^{e,short}, \tilde{\pi}_t, \tilde{\pi}_t^{short})$ are (1, 1, 0.5, 2, 1), respectively. κ_ϵ takes the value 100. Finally, for the prior distribution $p(\mu^\pi)$ of the bias term μ^π, we use $\underline{\mu}^\pi = 1$ and , $\underline{\sigma}_{\mu^\pi}^2 = 1$.

Appendix B. Summary of the method of Holston, Laubach, and Williams (2023)

In this appendix, we outline the application of the method of the Holston, Laubach and Williams (2023, hereafter HLW) to Japan. The original models, Laubach and Williams (2003) and HLW (2017), have long been benchmarks for semi-structural models, but problems arose during the COVID-19 pandemic when input data deviated significantly from the model assumptions, increasing uncertainty in the estimates. The HLW (2023) method addresses these problems by (1) taking into account the impact of persistent supply constraints associated with the COVID-19 pandemic on potential output and (2) providing flexibility in the variance of the output gap and inflation rate after COVID-19. The estimation methodology explained below is based on Fujiwara et al. (2016), which adapts the methodology of Laubach and Williams (2003) to Japan, and is modified to cope with COVID-19 by referring to the methodology of HLW (2023).

This method can be described as a maximum likelihood estimation of the state space representation of the IS curve and Phillips curve as the observation equation and the dynamic equations of state variables such as r^* and the potential GDP as the state equation. Specifically, the observation equation is given by

$$\tilde{y}_t^{adj} \quad = \quad a_{y,1}\tilde{y}_{t-1}^{adj} + a_{y,2}\tilde{y}_{t-2}^{adj} + a_r \frac{r_{t-1} - r_{t-1}^{*} + r_{t-2} - r_{t-2}^{*}}{2} + \varepsilon_t^{\tilde{y}},\ \varepsilon_t^{\tilde{y}} \sim N\ (0, {\sigma_{\tilde{y}}}^2) \qquad \text{(A2-1)}$$

$$\pi_t \quad = \quad \beta_1 \pi_{t-1} + \beta_2 \sum_{i=2}^{4} \frac{\pi_{t-i}}{3} + (1 - \beta_1 - \beta_2) \sum_{i=5}^{8} \frac{\pi_{t-i}}{4} + \beta_3 (y_{t-1} - y_{t-1}^{*}) \qquad \text{(A2-2)}$$

$$+\beta_4 (\pi_t^l - \pi_t) + \beta_5 (\pi_t^o - \pi_t) + \varepsilon_t^{\pi}, \varepsilon_t^{\pi} \sim N\ (0, {\sigma_{\pi}}^2)$$

where $\tilde{y}_t^{adj}$ is the supply constraint-adjusted output gap, r_t is the real short-term interest rate, π_t is the inflation rate, y_t is GDP, y_t^* is the potential GDP, π_t^l is the year-on-year change in import goods price, and π_t^o is the year-on-year oil price change. Next, one of the key points of the HLW (2023) methodology, the supply constraint-adjusted output gap $\tilde{y}_t^{adj}$ assumes the following relationship between y_t and y_t^*.

$$\tilde{y}_t^{adj} = y_t - \ (y_t^* + \phi d_t) \qquad \text{(A2-3)}$$

where d_t represents the negative supply constraint shock. Thus, the Phillips curve expressed in (A2-2) is formulated based on the output gap including the effect of the negative supply constraint shock d_t $(\equiv y_t - y_t^*)$, while the IS curve expressed in (A2-1) is the supply constraint adjusted output gap $\tilde{y}_t^{adj}$ netting of the effect of supply constraint, d_t. In other words, we assume that the decline in GDP and the worsening of the output gap in the COVID-19 pandemic do not affect the estimated value of r^* insofar as it is considered to have been caused by supply constraint.

HLW (2023) also attempts to capture the effect of COVID-19 by giving flexibility to the error terms in the IS and Phillips curves. That is, assuming each error term follows a normal distribution like $\varepsilon_t^{\tilde{y}} \sim N\ (0, \left(\kappa_t \sigma_{\tilde{y}}\right)^2)$ and $\varepsilon_t^{\pi} \sim N\ (0, (\kappa_t \sigma_\pi)^2)$, while the time-varying parameter κ_t

$$\kappa_t = \begin{cases} \kappa_{2020}, & 2020{:}Q2 \leq t \leq 2020{:}Q4 \\ \kappa_{2021}, & 2021{:}Q1 \leq t \leq 2021{:}Q4 \\ \kappa_{2022}, & 2022{:}Q1 \leq t \leq 2022{:}Q4 \\ 1, & otherwise \end{cases} \tag{A2-4}$$

given in the form of κ_{2020}, thereby reducing the impact of the large reduction in GDP and other factors caused by COVID-19. This method can be applied in the estimation of r^* for Japan, but the value of κ_{2020} was less than 1 when we estimated r^* with the input data described below. This is contrary to the results of HLW (2023), which considers that changes in the supply-constraint-adjusted output gap $\tilde{y}_t^{adj}$ and the inflation rate π_t that cannot be captured by the IS curve and Phillips curve have increased since the COVID-19 pandemic. In other words, the adjustment by negative supply constraint shock $\mathrm{d_t}$ alone sufficiently improve the interpretability of the IS curve and Phillips curve in Japan. Based on these results, the estimation method in this paper assumes $\kappa_t = 1$ over the entire period.

The input data used are basically the same as in Fujiwara et al. (2016). The output is real GDP (in log terms), the inflation rate is the year-on-year change in the consumer price index (all items excluding fresh food, all items excluding seasonal products before 1969, seasonally adjusted and consumption tax adjusted), the import price inflation rate is the year-on-year change in the import price index, the oil price inflation rate is the year-on-year change in oil prices based on trade statistics, and the real short-term interest rate is calculated by the unsecured overnight call rate (prior to the 1985/2Q, estimated from the secured call rate) minus the inflation expectation which is obtained by the AR(3) model's projection.

HLW (2023) also uses the "COVID-19 Stringency Index" from the Oxford COVID-19 Government Response Tracker (OxCGRT) for the supply constraint shock d_t in Eq. (A2-3). The movement of the index in Japan shows that the peak was in 2021/3Q, which does not coincide with 2020/2Q, when Japan's GDP deteriorated the most in the COVID-19 pandemic. This suggests that, at least in Japan, the index may not be appropriate as a proxy for supply constraints in Japan. Therefore, we use as an alternative indicator the index for

Retail and recreation in the Google COVID-19 Community Mobility Report, which captured the impact of the COVID-19 pandemic and bottomed out in 2020/2Q.[21]

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[21] There are many possible candidates for variables representing supply constraints. For our estimation, we compared several specifications, including the COVID-19 Stringency Index, Google Mobility Report (for each index category - Retail and recreation, etc.), and Google Trends (the number of times "supply constraints" was searched from Japan). As a result, the Google Mobility Report (Retail and recreation) was selected as the estimated ϕ was the largest, and it was seen to more accurately reflect supply constraints under the spread of COVID-19.

Appendix C. Summary of the methods of Imakubo, Kojima, and Nakajima (2015) and Nakajima et al. (2023) (Natural yield curve model)

This section outlines the methods of Imakubo, Kojima, and Nakajima (2015) and Nakajima et al. (2023). Similar to Laubach and Williams (2003), these methods capture business cycles based on the IS curve. However, while traditional models find a relationship between the interest rate gap (defined as the difference between the short-term real interest rate and the natural rate of interest) and the output gap in the IS curve, Imakubo, Kojima, and Nakajima (2015) and Nakajima et al. (2023) extend the concept of r^* to the longer maturity. In other words, they define the shape of the economic-neutral yield curve as the equilibrium yield curve, which corresponds to the short-term r^*, and consider the IS curve based on the output gap and the yield curve gap given by the difference between the actual and equilibrium yield curve.

Estimation is performed in two steps according to Brzoza-Brzezina and Kotłowski (2014): (1) estimating the factors of the real yield curve based on the dynamic Nelson-Siegel decomposition, and (2) estimating the equilibrium yield curve.

For the first step, consider the following equation:

$$r_{\tau,t} = L_t + S_t \frac{1-e^{-\lambda\tau}}{\lambda\tau} + C_t \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) + \varepsilon_{\tau,t}, \ \ \varepsilon_{\tau,t} \sim N \ \left(0, {\sigma_{\varepsilon_\tau}}^2\right) \tag{A3-1}$$

$$\begin{bmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{bmatrix} + \begin{bmatrix} \xi_t^L \\ \xi_t^S \\ \xi_t^C \end{bmatrix}, \xi_t^i \sim N\left(0, {\sigma_{\xi^i}}^2\right) for \ \ i = L, S, C \tag{A3-2}$$

where $r_{\tau,t}$ is the real interest rate for maturity τ, λ is a parameter of the Nelson-Siegel model, L_t is the level, S_t is the slope, and C_t is the curvature factor. Eq. (A3-1) is the observation equation for the real yield curve, and Eq. (A3-2) is the transition equation for each factor. In the first stage, each factor is estimated using the maximum likelihood method with a Kalman filter.

In the subsequent second step, the equilibrium yield curve is estimated from the IS curve extended in the direction of the maturity, taking each factor as a given. The specific formula form is as follows.

$$\begin{bmatrix} y_t - y_t^* \\ L_t \\ S_t \\ C_t \end{bmatrix} = \begin{bmatrix} a_y & b_L & b_S & b_C \\ 0 & a_L & 0 & 0 \\ 0 & 0 & a_S & 0 \\ 0 & 0 & 0 & a_C \end{bmatrix} \begin{bmatrix} y_{t-1} - y_t^* \\ L_{t-1} \\ S_{t-1} \\ C_{t-1} \end{bmatrix} + \begin{bmatrix} b_L & b_S & b_C \\ 1 - a_L & 0 & 0 \\ 0 & 1 - a_S & 0 \\ 0 & 0 & 1 - a_C \end{bmatrix} \begin{bmatrix} L_t^* \\ S_t^* \\ C_t^* \end{bmatrix} \tag{A3-3}$$

$$+ \begin{bmatrix} 1 & 0 & 0 & 0 \\ g_{yL} & 1 & 0 & 0 \\ g_{yS} & 0 & 1 & 0 \\ g_{yC} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t^y \\ \varepsilon_t^L \\ \varepsilon_t^S \\ \varepsilon_t^C \end{bmatrix}$$

where $y_t - y_t^*$ is the output gap, and L_t^*, S_t^*, C_t^* are the level, slope, and curvature factors of the equilibrium yield curve, respectively. The first line of Eq. (A3-3) defines the IS curve extended in the direction of the maturity and the other lines define the dynamics of the real yield curve. In addition, given $\phi(\tau)$ defined as parameters representing the difference in the sensitivity of the output gap to the interest rate gap in each maturity ($g_{\tau,t} \equiv r_{\tau,t} - r_{\tau,t}^*$), the parameters b_L, b_S, b_C satisfy the following conditions.

$$\frac{b_L}{b} = \int_0^T \phi(\tau) d\tau \tag{A3-4}$$

$$\frac{b_S}{b} = \int_0^T \left\{ \phi(\tau) \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right\} d\tau \tag{A3-5}$$

$$\frac{b_C}{b} = \int_0^T \left\{ \phi(\tau) \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) \right\} d\tau \tag{A3-6}$$

Note that $\phi(\tau) \geq 0, \int_0^T \phi\ (\tau) d\tau \equiv 1$ for standardization. Next, the dynamics of each factor of the equilibrium yield curve are given by the relationship with the potential growth rate ($g_{y,t}^* \equiv y_t^* - y_{t-1}^*$), but different formulations are possible depending on whether we correspond each factor to the rate of change in the potential growth rate or its level.

$$\begin{bmatrix} L_t^* \\ S_t^* \\ C_t^* \end{bmatrix} = \begin{bmatrix} L_{t-1}^* \\ S_{t-1}^* \\ C_{t-1}^* \end{bmatrix} + \begin{bmatrix} h_{yL} \\ h_{yS} \\ h_{yC} \end{bmatrix} \left(g_{y,t}^* - g_{y,t-1}^* \right) + \begin{bmatrix} 1 & 0 & 0 \\ h_{LS} & 1 & 0 \\ h_{LC} & h_{SC} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t^{L^*} \\ \varepsilon_t^{S^*} \\ \varepsilon_t^{C^*} \end{bmatrix} \tag{A3-7}$$

$$\begin{bmatrix} L_t^* \\ S_t^* \\ C_t^* \end{bmatrix} = \begin{bmatrix} p_{L^*} & 0 & 0 \\ 0 & p_{S^*} & 0 \\ 0 & 0 & p_{C^*} \end{bmatrix} \begin{bmatrix} L_{t-1}^* \\ S_{t-1}^* \\ C_{t-1}^* \end{bmatrix} + \begin{bmatrix} h_{yL} \\ h_{yS} \\ h_{yC} \end{bmatrix} g_{y,t}^* + \begin{bmatrix} 1 & 0 & 0 \\ h_{LS} & 1 & 0 \\ h_{LC} & h_{SC} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t^{L^*} \\ \varepsilon_t^{S^*} \\ \varepsilon_t^{C^*} \end{bmatrix} \tag{A3-8}$$

The formulation corresponding to the rate of change (Eq. A3-7) is used in Imakubo, Kojima, and Nakajima (2015), while the formulation corresponding to the level (Eq. A3-8) is used in Nakajima et al. (2023).

The data used in the estimation can be broadly classified into nominal zero-coupon interest rate, inflation expectation, output gap, and potential growth rate. The real zero-coupon rate, which is the input data for the estimation, is a series of the nominal zero coupon rate minus the inflation forecast from “Consensus Forecast”. Since the period covered by the survey of inflation expectations is limited to 10 years ahead, expectations beyond 10 years are assumed to be the same as the forecasts for 10 years ahead. In addition, since the survey is conducted semiannually, quartiles are derived using linear interpolation. The real zero-coupon interest rates used in the estimation have annual maturities of 1, 2, 3, 7, 10, and 20 years, and the output gap and the potential growth rate are estimated by the Research and Statistics Department of the Bank of Japan. Finally, the sample period is from 1992/3Q to 2023/1Q.